This is a digital copy of a book that was preserved for generations on library shelves before it was carefully scanned by Google as part of a project to make the world's books discoverable online.

It has survived long enough for the copyright to expire and the book to enter the public domain. A public domain book is one that was never subject to copyright or whose legal copyright term has expired. Whether a book is in the public domain may vary country to country. Public domain books are our gateways to the past, representing a wealth of history, culture and knowledge that's often difficult to discover.

Marks, notations and other marginalia present in the original volume will appear in this file - a reminder of this book's long journey from the publisher to a library and finally to you.

Usage guidelines

Google is proud to partner with libraries to digitize public domain materials and make them widely accessible. Public domain books belong to the public and we are merely their custodians. Nevertheless, this work is expensive, so in order to keep providing this resource, we have taken steps to prevent abuse by commercial parties, including placing technical restrictions on automated querying.

We also ask that you:

+ *Make non-commercial use of the files* We designed Google Book Search for use by individuals, and we request that you use these files for personal, non-commercial purposes.

+ *Refrain from automated querying* Do not send automated queries of any sort to Google's system: If you are conducting research on machine translation, optical character recognition or other areas where access to a large amount of text is helpful, please contact us. We encourage the use of public domain materials for these purposes and may be able to help.

+ *Maintain attribution* The Google "watermark" you see on each file is essential for informing people about this project and helping them find additional materials through Google Book Search. Please do not remove it.

+ *Keep it legal* Whatever your use, remember that you are responsible for ensuring that what you are doing is legal. Do not assume that just because we believe a book is in the public domain for users in the United States, that the work is also in the public domain for users in other countries. Whether a book is still in copyright varies from country to country, and we can't offer guidance on whether any specific use of any specific book is allowed. Please do not assume that a book's appearance in Google Book Search means it can be used in any manner anywhere in the world. Copyright infringement liability can be quite severe.

About Google Book Search

Google's mission is to organize the world's information and to make it universally accessible and useful. Google Book Search helps readers discover the world's books while helping authors and publishers reach new audiences. You can search through the full text of this book on the web at `http://books.google.com/`

PLANE GEOMETRY

[illegible]

Educ T 149.16.970

HARVARD COLLEGE
LIBRARY

GIFT OF THE
GRADUATE SCHOOL
OF EDUCATION

3 2044 097 046 510

TWENTIETH CENTURY TEXT-BOOKS

TWENTIETH CENTURY TEXT-BOOKS

THE YOUNG AND JACKSON ALGEBRA SERIES

A High School Algebra
x, 508 pages, $1.15.

A High School Algebra, Part I
vii, 294 pages, 95 cents.

A High School Algebra, Part II
viii, 216 pages, 70 cents.

Elementary Algebra
xii, 442 pages, $1.12.

A First Course in Elementary Algebra
ix, 294 pages, 95 cents.

A Second Course in Elementary Algebra
vii, 206 pages, 70 cents.

All of the above texts may be had with or without answers.

D. APPLETON AND COMPANY
NEW YORK CHICAGO

TWENTIETH CENTURY TEXT-BOOKS

PLANE GEOMETRY

BY

J. W. A. YOUNG, PH.D.
ASSOCIATE PROFESSOR OF THE PEDAGOGY OF MATHEMATICS
THE UNIVERSITY OF CHICAGO

AND

LAMBERT L. JACKSON, PH.D.
ASSISTANT SUPERINTENDENT OF SCHOOLS
MONTCLAIR, NEW JERSEY

D. APPLETON AND COMPANY
NEW YORK CHICAGO

Educ T 149.16.970

HARVARD COLLEGE LIBRARY
GIFT OF THE
GRADUATE SCHOOL OF EDUCATION
May 8, 1930

Copyright, 1916, by
D. APPLETON AND COMPANY.

PREFACE

THE facts of geometry are unchangeable. The experience of the past has determined what body of these facts is suitable for secondary school work, both for teaching the geometric type of reasoning and for application in the arts and sciences. Therefore, a book that would avoid the bizarre and the erratic must profit by the experience of the past and confine its attempt at novelty to the field of method. More skillful presentations and more effectual ways of reaching the chief ends of the teaching of geometry are always possible.

Accordingly, the authors of this work have given full consideration to the history of geometry and to the views of professional committees of recent date, and they here present in the form of definitions, propositions, and corollaries, that minimum of essentials which has become standard. Furthermore, every effort has been made by means of phraseology, sequence, and typographic arrangement, to secure the maximum of lucidity.

The time is past when it was necessary to argue that in studying geometry the pupil should not merely learn demonstrations but also do some demonstrating himself. Accordingly, frequent and extensive sets of exercises furnish much occasion for actual demonstration on the part of the pupil. But the demonstrations of the basic theorems have been given in full or in sufficiently detailed outline, both in order that the pupil may throughout the subject have authoritative models for his guidance, and also that the building up of the main body of the subject should not be left to the hazards of the more or less complete success of the pupil in mastering the methods of geometry in content and form. Preparatory work has been given whenever desirable to facilitate the pupil's approach to the proposition in hand. Special attention has been given to the applications of the subject and the aim has been to show, by numerous simple instances, the wide range of usefulness of geometric truth. The sets of exercises of all the various types have been made as extensive as space will permit in order to furnish an ample supply for the teacher's selection according to the time available and the needs of the class.

THE AUTHORS

CONTENTS

PAGE

ELEMENTARY NOTIONS 1

BOOK I. ANGLES, TRIANGLES, AND QUADRILATERALS . . 28

BOOK II. CIRCLES 108
  Properties of the Circle 108
  Constructions 113

BOOK III. MEASURES, PROPORTION, AND SIMILAR FIGURES . 170
  Ratio and Proportion 172
  Similarity 187

BOOK IV. MEASUREMENT OF POLYGONS 214
  Nature of Measurement 214
  Metrical Properties 216

BOOK V. REGULAR POLYGONS AND CIRCLES 250
  Properties of Regular Polygons 250
  Measurement of Circles 253

SUPPLEMENT 281
  Symmetry 281
  Maxima and Minima 283
  Signed Magnitudes 293
  Limits 299
  Formulas 306

INDEX 309

SYMBOLS AND ABBREVIATIONS

= equals, or is equal to.
> is greater than.
< is less than.
≠ is not equal to.
‖ parallel, or is parallel to.
⊥ perpendicular, or is perpendicular to.
∠ angle.
∟ right angle.
△ triangle.
▱ parallelogram.
□ rectangle.
⊙ circle.
$\overline{AB}^2$ square with side AB; or as a number, the square of the length of AB.
st. straight.
rt. right.

π pi.
∵ since.
∴ therefore; hence.
··· and so on.
Alt. alternate.
Ax. axiom.
Const. construction.
Cor. corollary.
Def. definition.
Ex. exercise.
Ext. exterior.
Fig. figure.
Hyp. hypothesis.
Int. interior.
Post. postulate.
Prob. problem.
Prop. proposition.
Sec. section.

The plural form of certain symbols is also used. Thus, ‖s, ∡, ⟁ or △'s, ⊚ or ⊙'s.

The symbols +, −, ×, ·, :, ÷, √ are used as in algebra.

Both angles and arcs are expressed in *degrees*, *minutes*, and *seconds*. Thus, 5° 10′ 45″ may, according to the context, mean either an angle or an arc.

There is no generally accepted symbol for "is congruent to," hence these words themselves are used in this book. (Some writers prefer ≌ or ≅, and some use ≡, but the sign of equality is more commonly employed.)

Q.E.D. is an abbreviation commonly used in geometry for the Latin words *quod erat demonstrandum*, "which was to be proved."

Q.E.F. stands for *quod erat faciendum*, "which was to be done."

PLANE GEOMETRY

ELEMENTARY NOTIONS

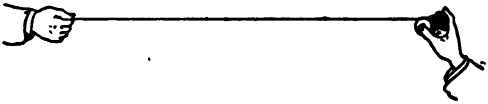
1. Two Aspects of Geometry. Arithmetic and algebra contain two kinds of material, definitions and processes on the one hand and practical applications on the other. The applications of arithmetic pertain largely to everyday business practice, while the applications of algebra pertain more to the problems of science. Geometry, likewise, contains two types of work, — the theoretical and the practical, — and, like algebra, finds the greater part of its applications in science, although its processes of measurement often enter problems of business.

Early Practical Geometry. Among the Egyptians, who did the earliest geometric work of which we have a record, geometry meant a system of surveying by which, after each overflow of the Nile, the obliterated boundary lines were reëstablished. This required a knowledge of the properties of certain figures, for example, the right triangle. The word "geometry," which comes from the Greek and means "earth measurement" or "land measurement" is thus descriptive of the origin of the science.

Early Theoretical Geometry. In marked contrast with the practical work of the Egyptians, the Greeks gave almost exclusive attention to the theoretical study of the properties of figures, and developed a system of *proof* by strict reasoning, which is still used at the present time.

Both Aspects in the Present Work. In what follows, we shall treat geometry from both of these aspects; on the one hand, establishing certain theoretical properties of figures, and on the other, finding out their principal practical uses.

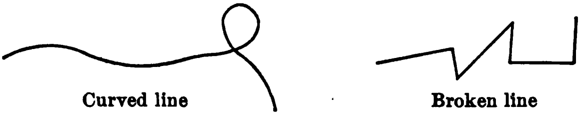
2. Straight Line. The ideas of a line and of a straight line are familiar to every one. A clear image of a line is furnished by a tensely drawn thread.



3. Geometric Line. A geometric line is not thought of as consisting of matter; it is only the image of the position where a material line might be.

4. Broken Line. A line that is not straight, but that is composed entirely of straight parts, is called a **broken line.**

5. Curved Line. A line of which no part is straight is called a **curved line**, or, more briefly, a **curve.**



A line which is in part straight and in part curved is called a **mixed line.**

What sort of lines are each of the following letters: A, U, I, O, E, C, D?

6. In geometry straight lines are thought of as unterminated, that is, as extending without end in both directions.

7. Intersect. When two straight lines cross they are said to **intersect.**

8. Point. The place where the two straight lines intersect is called a **point.**

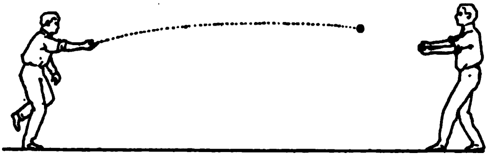
Each straight line is said to "pass through" the point. The point is said to "lie on" each straight line.

The intersections of straight lines with curves, and the intersections of curves with each other are called points.

Points may also be thought of otherwise than as the intersections of lines. For example, the image of a point may be furnished by a small grain of sand.

If a drop of rain is thought of as a point and this drop falls straight down, the path of the drop may be thought of as a straight line.

If a ball is thought of as a point, the path of the ball, when thrown by a player, may be thought of as a curved line.

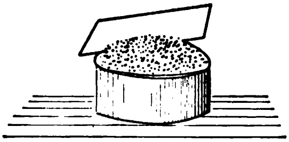


9. **Surface.** The idea of a flat surface is familiar to every one. An image of such a surface is furnished by the ceiling of a room, or by the surface of a well made blackboard.

10. **Plane.** A flat surface is called a **plane.**

Planes, like straight lines, are thought of as unterminated in extent.

If the edge of a straight board is thought of as a straight line, and this board is moved over a measure heaped up with grain, the surface thus made may be thought of as a plane.

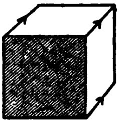


This example illustrates the possibility of regarding a plane as produced (or "generated") by a moving straight line. We are thus led to the following characteristic property of a plane: *An unlimited surface of such a character that a straight line joining any two of its points lies entirely in the surface, is a plane.*

11. Geometric Solid. A limited portion of space is called a **geometric solid.**

A portion of a plane, when suitably moved from one position to another, may be thought of as generating a geometric solid.

Thus, if the square portion of a plane in the front of the figure is moved straight back in the direction of the arrows, to a distance equal to the side of the square, a cube is generated.



12. Surface. The boundary between a geometric solid and the space around it is called the **surface** of the solid.

The boundary between a plane surface and the rest of the plane around it is a *line.* The boundary between a portion of a line and the rest of the line is a *point.*

13. Geometric Figures. Points, lines, surfaces, solids, or any of their combinations, are called **geometric figures.**

14. Plane Figures. When all of a figure is in one plane, it is called a **plane figure.**

15. Geometry. Geometry is the science which treats of geometric figures.

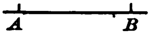
Plane geometry treats of plane figures; solid geometry treats of figures that do not lie wholly in one plane.

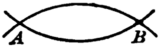
16. Notation for Points. It is customary to name points by single letters, as A, or P.

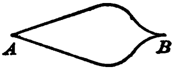
17. If any two points are given, we may think of them as lying on a straight line.

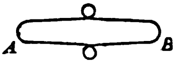
If the two points are marked by two pins, and if a thread is stretched taut touching both pins, the thread furnishes the image of the straight line passing through the two points. If another thread is so used it coincides with the first one.

It is clear that only one straight line can pass through two points, and that this is not true for lines of a given shape that are not straight.



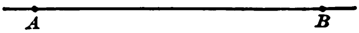






18. Thus, two points completely fix the position of a straight line; this is expressed briefly by saying that *two points determine a straight line.*

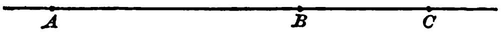
19. Notation for Lines. A straight line may be named by a single letter, as the straight line l. To avoid confusion, single small letters are usually used to designate lines, and single capital letters to designate points. A straight line may also be named by two points on it, as the straight line AB.



When there is no danger of confusion, the word "straight" is often omitted, and we speak of "the line AB" meaning "the straight line AB."

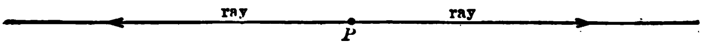
20. Segment. The portion of a straight line between two of its points is called a **segment** of the line.

If the points are called A and B, the segment is called AB. Thus, AB, BC, and AC are segments.



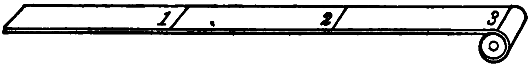
Strictly taken, "the line AB" means the unterminated straight line through these two points, but where there is no danger of confusion, segments are frequently called "lines."

21. Ray. Any point on a straight line divides it into two parts often called **rays**, each of which extends without end in one direction from the point.



22. Equal Segments. The segments are **equal** when they can be made to fit each other, or to coincide exactly.

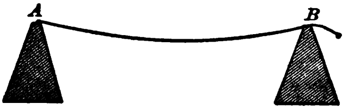
Thus, if a tape is placed on a ribbon so as to coincide with it, as in the illustration, each is a yard in length and the edges represent equal straight lines.



23. The **symbol for equality** is the same as that used in arithmetic.

24. Preparatory.

1. If a string is fastened at *A* and lies in any position not straight between two points *A* and *B*, it can be made more nearly straight by pulling a certain length of string beyond *B*. That is, it takes less string to connect the points by a straight path than by any other.



2. If a straight wire is passed through two holes, as shown in Fig. 1, and if it is turned without gliding, it will not alter its position; but a curved line will alter its position if so turned, as shown in Fig. 2.

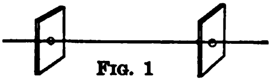


FIG. 1

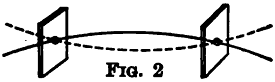


FIG. 2

3. If a straight wire is moved along through two closely fitting holes, as in Fig. 3, the individual points are moved, but the position of the wire as a whole, regarded as an unending straight line, remains the same.

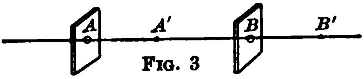
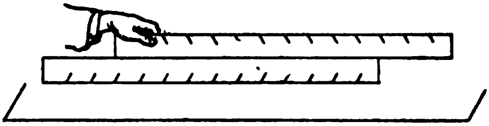


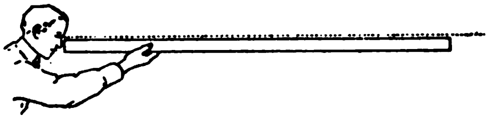
FIG. 3

This property may be used to test edges supposed to be straight. If they are straight it must be possible to bring them into complete contact and to slide them along each other



without breaking the contact (except where one projects beyond the other).

When we "sight" along an edge to see whether or not it is straight, we are really comparing the edge with the path of a beam of light, knowing that light travels in practically straight lines.



25. Properties of Straight Lines. We have thus seen three important properties of the straight line.

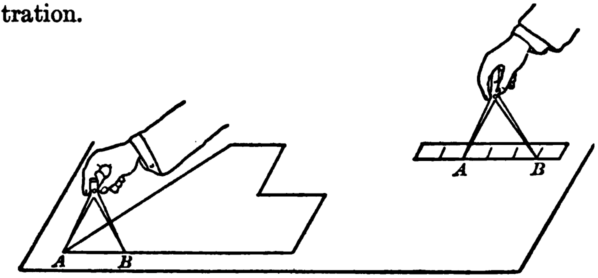
1. *The straight line is the shortest line between two points.*

2. *If two points of a straight line are fixed and the straight line is rotated, the position of the straight line is not altered.*

3. *If two straight lines are in complete contact, either can glide along the other for any distance without breaking the contact.*

The pupil should feel that these statements are fully verified by illustration and experiment.

26. Laying off Segments. Segments equal to given segments may be marked off by means of compasses, as indicated in the illustration.

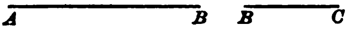


Similarly, a given segment may be laid off one or more times on a longer segment.

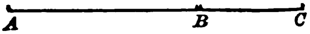
27. Bisector. If a line segment is divided by a point into two equal segments, it is said to be **bisected**, and the point is called the **bisector** of the segment. There is only one such point.

Similarly, if the line segment is divided by two points into three equal parts, it is said to be **trisected**, and the points are called the **trisection points** of the segment.

28. Addition of Segments. When two segments are placed end to end in a straight line so as not to overlap, the segment formed is called the *sum* of the given segments.



Two segments



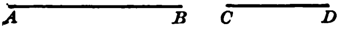
Their sum

The sum is indicated by the sign + connecting the symbols for the segments. Thus,

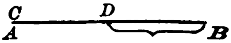
$$AB + BC = AC.$$

The sum of three or more segments can be formed similarly.

29. Subtraction of Segments. When one of two unequal segments is placed upon the other so that a pair of end points coincide, the portion of the longer segment not covered by the shorter is called their *difference*.



Two segments

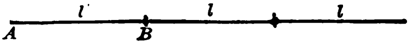


Their difference

The difference is indicated by the sign, –, connecting the symbols for the segments.

$$AB - CD = DB.$$

30. Multiplication of Segments. A segment may be multiplied by an integer by repeated addition.



$$AB + AB + AB = 3\,AB, \text{ or } l + l + l = 3\,l.$$

The absence of a sign indicates multiplication, as in algebra.

Whenever desirable, the dot or the cross or the signs of grouping may be used, as in algebra.

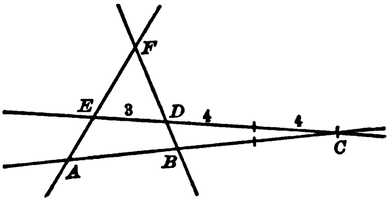
EXERCISES

In the figure, name:

1. Three pairs of points such that each pair determines the line *AB*.

2. Three pairs of points such that each pair determines the line *BF*.

3. The point of intersection of *AE* and *BD*. Of *AB* and *ED*.



Read a single segment equal to each of the following:

4. $ED + DC$.

5. $AB + BC$.

6. $BD + DF$.

7. $EC - DC$.

8. $AC - AB$.

9. $FD + DB$.

10. What is the length of segment *EC*?

11. Of what two segments is *AF* the sum?

12. Of what two segments is *DC* the difference?

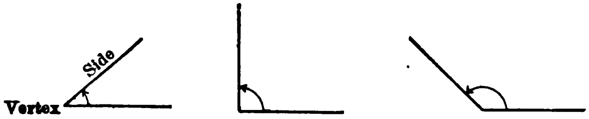
13. Draw two unequal straight line segments. Draw their sum; their difference.

14. Draw a segment that shall be four times a given segment.

15. Mark three points not lying in the same straight line. Draw all the straight lines determined by any two of these points. How many of these lines are there?

31. Angle. The figure formed by two rays extending from the same point is called an **angle.**

The two rays are called the *sides* of the angle, and their common point the *vertex.*



These figures represent angles differing in size.

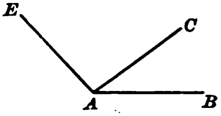
32. If one side of an angle is turned about the vertex until it takes the position of the other side, it is said to *generate* the angle.

33. Size of Angles. The size of an angle depends upon the amount of turning required to generate it.

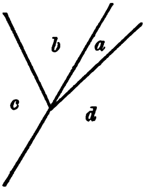
In the above figures, the amount of turning in the first angle at the left is less than that in the second, which is in turn less than that in the third. The first angle is the smallest and the third is the largest.

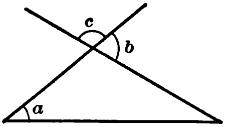
34. Plane Angles. There are other angles besides those formed by straight lines. Hence, to designate the latter, we may call them **plane angles**, but when there is no chance for a misunderstanding we shall omit the word "plane."

35. Notation for Angles. Angles are named either by the letter at the vertex, as *A*, or by a letter on each side, with the letter at the vertex between them, as *BAC*, or *BAE* in the figure.



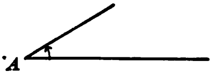
Angles are also named by a small letter or other character placed within the angle, as: *a*, *b*, *c*, *d*, in the figures.

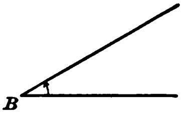


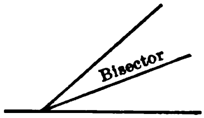


36. Equal Angles. Two angles are **equal** when they can be made to coincide, that is, when the vertex and the sides of one can be made to lie on the vertex and the sides of the other, without changing the size of either angle.

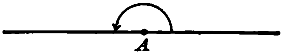
In the figure, angle *A* equals angle *B*. This can be seen by drawing angle *A* on tracing paper and placing the image on angle *B*.





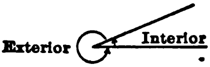


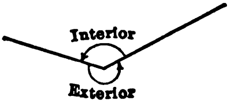
37. Bisector. If an angle is divided into two equal angles by a line through its vertex, the angle is said to be bisected, and the line is called the **bisector** of the angle. If an angle is divided into three equal parts it is **trisected.**

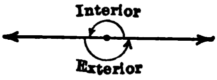


38. Straight Angle. The angle formed by the two rays into which any point of a straight line divides that line is called a **straight angle.**

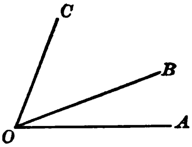
39. Exterior and Interior Angles. When two rays diverge from the same point two angles are really formed. When it is necessary to distinguish between them, one may be called the **interior** angle and the other the **exterior** angle.







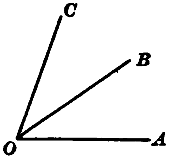
40. Reflex Angle. An angle that is greater than a straight angle but smaller than two straight angles is called a **reflex angle.**

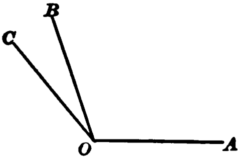


41. Adjacent Angles. Two angles that do not overlap and that have the same vertex and a common side are called **adjacent angles.**

Thus, angles AOB and BOC are adjacent angles, but angles BOC and AOC are not.

42. Addition of Angles. The angle formed by the outer sides of two adjacent angles is called the **sum** of the given angles.

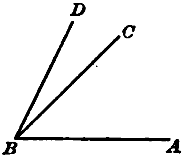




Thus,

$\angle AOC = \angle AOB + \angle BOC.$ $\angle AOC = \angle AOB + \angle BOC.$

43. If two unequal angles are so placed as to overlap and to have a common side and vertex, one angle will lie entirely within the other. This angle is called the *smaller* angle, and the other the *larger*.



Thus, angle ABC is smaller than angle ABD, and angle ABD is the larger.

The **symbols of inequality** are given in the table opposite to page 1.

Thus, in the figure $\angle ABD > \angle ABC$; $\angle CBA < \angle DBA$; $\angle CBD \neq ABC$.

44. Subtraction of Angles. When two overlapping angles have a common side and vertex, the angle formed by the other sides is called the *difference* of the angles.

The difference is denoted by the sign, —, connecting the symbol for the larger angle with that for the smaller angle.

Thus, in the figure of sec. 43 $\angle CBD = \angle ABD - \angle ABC$.

EXERCISES

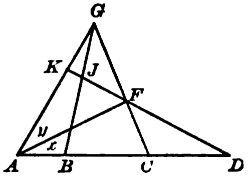
1. Read the angle x in three different ways.

2. Read the angle y in two different ways.

3. Read all of the angles at the vertex G.

4. Read all of the angles at the point F.

5. Name two angles that are larger than AFK.



6. Name two angles that are larger than AFG.

7. What points named in the figure lie on the sides of the angle DAG?

8. What points named in the figure lie in the interior of this angle?

9. Name two angles that are adjacent to angle AFK.

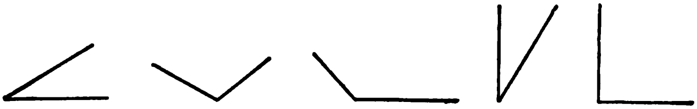
10. Name three pairs of adjacent angles.

Read the angles equal to:

11. $\angle AFK + \angle KFG$.   **13.** $\angle KFC - \angle KFA$.

12. $\angle CFA + \angle DFC$.   **14.** $\angle CFG - \angle AFK - \angle KFG$.

15. Draw angles that look like the following:



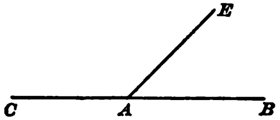
For each angle drawn, make an angle that you know to be larger; one that you know to be smaller.

16. Draw two adjacent angles. Point out their sum.

17. Draw two angles which overlap, and point out their difference.

45. Preparatory.

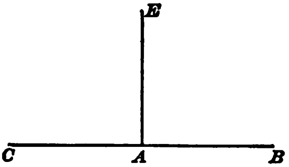
1. If a ray of a straight line revolves about its end point as vertex, it will at first form a small angle with one ray of the original line and a large angle with the other ray. For example, $\angle BAE$ and $\angle EAC$ in the figure.



2. The two angles together always make a straight angle.

3. If the line continues to revolve, angle *EAC* grows smaller and angle *BAE* grows correspondingly larger.

4. At *one* stage of the revolution the two angles will be equal and each will be half of a straight angle.



46. Right Angle. An angle that is half of a straight angle is called a **right angle.**

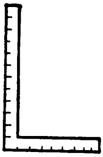
This may be stated in another way by saying that when a straight line is drawn from a point in another straight line, making adjacent angles equal, each angle is called a right angle.

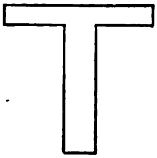
The symbol for a right angle is ∟, or $\underline{|R}$, and plural $\underline{|R}$'s.

47. Perpendicular. Two straight lines that form a right angle are said to be **perpendicular** to each other.

The symbol for "perpendicular" or "perpendicular to" is $\perp$.

A material corner whose sides form a right angle is called a "square corner." Carpenters draw right angles by means of a tool called a "square" consisting of two arms set at right angles to each other. Draftsmen use a "T-square" in which one arm projects on both sides of the other as in the letter T.


Carpenter's square


T-square

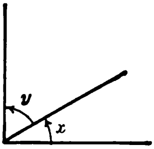
48. Acute Angle. An angle that is smaller than a right angle is called an **acute angle.**

49. Obtuse Angle. An angle that is larger than a right angle but smaller than a straight angle is called an **obtuse angle.**

50. Oblique Angle. An angle that is not a right angle is called an **oblique angle.**

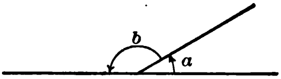
51. Complements. If the sum of two angles is a right angle, the angles are said to be **complementary** and each is called the **complement** of the other.

Thus, x and y are complementary angles.



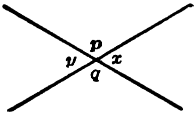
52. Supplements. If the sum of two angles is a straight angle, the angles are said to be **supplementary** and each is called the **supplement** of the other.

Thus, a and b are supplementary angles.



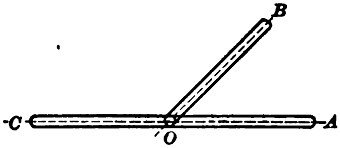
53. Vertical Angles. If the sides of an angle are extended beyond its vertex, the given angle and the angle between the extensions of the sides are called **vertical angles.**

Thus, x and y are vertical angles, also p and q. Two intersecting lines always form two pairs of vertical angles.



EXERCISES

1. The cut shows an instrument used for transferring angles. Read from it a pair of adjacent angles.

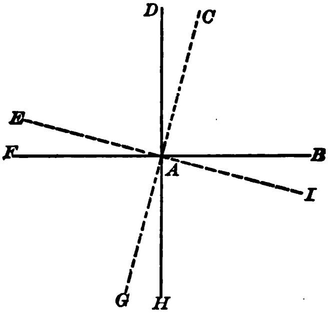


2. In the figure the solid lines intersect at right angles; the dotted lines also intersect at right angles. Read two right angles having DA as one side; read two acute angles having AC as a side.

3. Read two acute angles having GA as a side. Also two obtuse angles.

4. Name three pairs of complementary angles.

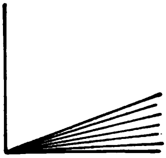
5. Name three pairs of supplementary angles.



6. Name two angles each of which is a complement of angle BAC.

7. Name two angles each of which is a supplement of angle DAE.

54. Measurement of Angles. For angles the unit of measure must be an angle. The right angle might be taken as a standard unit, but for convenience a much smaller angle has been chosen.



55. Degree. The customary unit angle is $\frac{1}{90}$ of a right angle, and is called a **degree.**

The symbol for the word "degree" or "degrees" is (°).

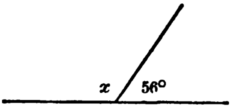
Thus, 87 degrees is written 87°.

To obtain a smaller unit, the degree is subdivided into sixty equal angles, each called one **minute**; symbol (′).

The minute is subdivided into sixty equal angles, each called one **second**; symbol (″).

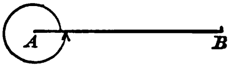
56. According to Sec. 55, a right angle contains 90° and a straight angle contains 180°.

The angle x in the figure must contain 180° – 56°, or 124°.



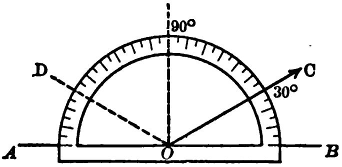
57. Perigon. The angle generated (Sec. 32) by a ray revolving about its end point until it returns to its original position is called a **perigon.**

Accordingly, a perigon is equal to four right angles, or 360°.



58. The Protractor. The protractor is an instrument used to measure or to draw angles approximately. Its nature is sufficiently clear from the illustration. To use the protractor the point O is placed at the vertex of the angle, and the line OB along one side of the angle. Then the number of degrees in the angle is shown where the other side, OC, cuts the scale.

For example, the angle BOC in the figure is one of 30°.

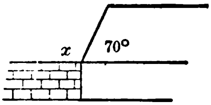


In a similar way, the protractor can be used to draw a line making any desired angle (less than 180°) with the line OB.

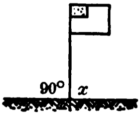
EXERCISES

State the number of degrees in x in each figure.

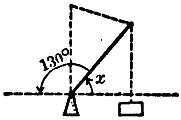
1.



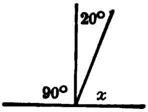
2.



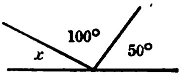
3.



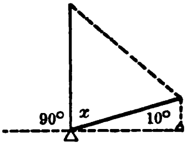
4.



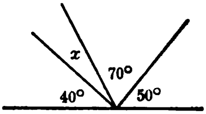
5.



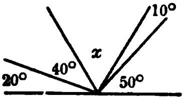
6.



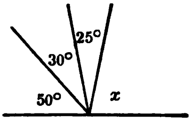
7.



8.

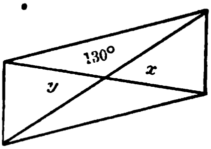


9.

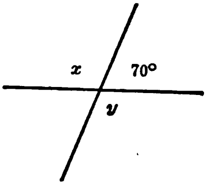


State the number of degrees in each unknown angle in each figure:

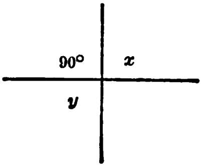
10.



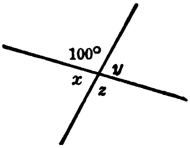
11.



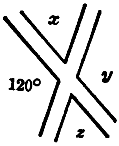
12.



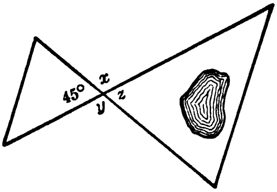
13.



14.



15.

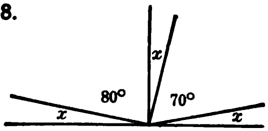


16. State the complement of each of the following angles: 30°; 70°; 14°.

17. State the supplement of each of the following angles: 20°; 100°; 160°; 85°.

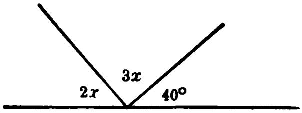
Find the number of degrees in the unknown angles of each figure:

18.



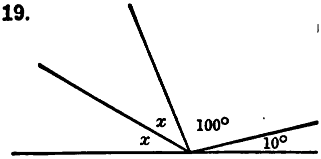
Thus, $3\,x^\circ + 80^\circ + 70^\circ = 180^\circ$.

20.

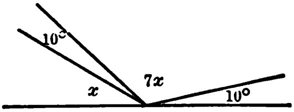


Thus, $2\,x^\circ + 3\,x^\circ + 40^\circ = 180^\circ$.

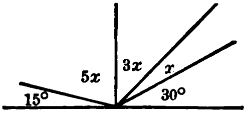
19.



21.



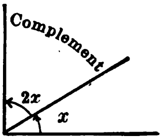
22.



23. Find the complement of: 17° 14′; 29° 8′; 54° 3′ 12″.

24. Find the supplement of: 18° 27′; 106° 29′ 14″; 143° 0′ 52″.

25. Find the number of degrees in an angle which is half its complement.



Suggestion. Let x = the size of the angle,
then $2\,x$ = the size of its complement.

Then $2\,x + x = 90$, by the definition of the complement.

Solve this equation for x.

Find the number of degrees in an angle which is:

26. Four times its supplement.

27. 16° more than its complement.

28. 10° less than its supplement.

29. 20° more than four times its complement.

30. 30° less than one fifth of its supplement.

31. Show that if the sum of two adjacent angles is a straight angle, their bisectors form a right angle.

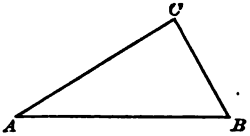
32. Show that if the bisectors of two adjacent angles do not form a right angle, the exterior sides of the angles do not form one straight line.

33. Show that the perpendicular to the bisector of an angle at its vertex bisects the adjacent supplementary angle.

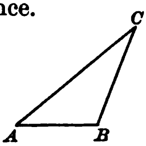
59. Triangle. A plane surface bounded by three straight lines is called a **triangle.**

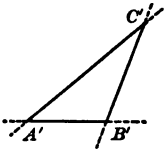
The lines are called the **sides** of the triangle, and the points where they meet, or intersect, are called its **vertices.**

Thus, in the figure, *AB*, *BC* and *CA* are the three sides of the triangle *ABC*, and *A*, *B* and *C* are its three vertices.



Usually only the segments between the vertices are regarded as the sides, but any or all of the sides may be extended to any desired distance.





60. Notation of Triangles. Triangles are named by reading the letters at the vertices.

Thus, in the above figure, the first triangle is read "Triangle *ABC*," the second, "Triangle *A* prime, *B* prime, *C* prime."

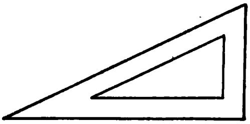
The symbol △ is often used for the word "triangle."

61. Perimeter. The distance around a triangle, or the sum of its three sides, is called its **perimeter.**

62. Right Triangle. A triangle one of whose angles is a right angle is called a **right-angled triangle**, or simply a **right triangle.**

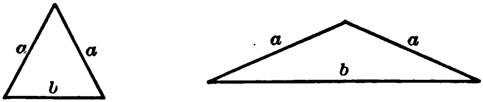
The side opposite the right angle is called the *hypotenuse,* the other two sides are called the *legs* of the right triangle.

The figure shows a right triangle made of wood, used in drawing.



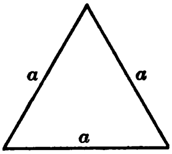
63. Isosceles Triangles. If two sides of a triangle are equal the triangle is called **isosceles.**

The two equal sides are called the *legs,* the third side is called the *base* of the isosceles triangle.



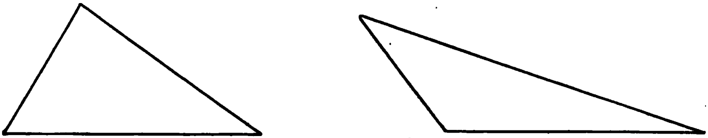
Isosceles triangles

64. Equilateral Triangle. If the three sides of a triangle are equal, the triangle is called **equilateral.**



Equilateral triangle

65. Scalene Triangle. If no two sides are equal, the triangle is called a **scalene triangle.**



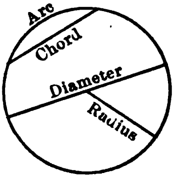
Scalene triangles

66. Equal Triangles. Two triangles that can be made to coincide are said to be **equal, or congruent.**

The coinciding ("corresponding") sides and angles must then also be equal.

67. Circle. A closed plane curve every point of which is equidistant from a fixed point of the plane is called a **circle.**

68. Center. The fixed point is called the **center** of the circle.



69. Radius. Any straight line from the center to the curve is called a **radius.**

70. Diameter. Any line through the center and terminated at both ends by the curve, is called a **diameter.**

71. Arc. Any part of the curve is called an **arc.**

72. Chord. The straight line segment connecting the end points of an arc is called the **chord** of the arc.

The chord is said to *subtend* the arc.

73. Area. The portion of plane inclosed by the circle is the **area** of the circle.

74. Circumference. The length of the curve is called the **circumference** of the circle, and this word is often used to designate the curve itself.

THE DRAWING OF FIGURES

75. Instruments Used. The constructions of plane geometry are arbitrarily limited to those which can be made with a ruler and compasses.

Instead of compasses, any apparatus for drawing a circle may be used, for example, a string. Of course the more accurate the instrument, the more satisfactory will be the results.

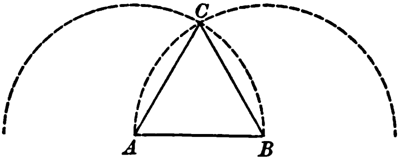
The following examples are given merely for the purpose of illustration. Formal constructions will be given later where their proofs can be explained.

EXERCISES

1. Draw two circles each with a radius of two inches and having their centers one inch apart. In how many points do they intersect?

2. Draw two circles each with a radius of one inch and having their centers two inches apart. Do the circles intersect?

3. Draw an equilateral triangle whose sides shall each equal a given line segment.



Construction.

1. Let AB be the given segment.

It is required to find a point C such that $CA = CB = AB$. This can be done with a pair of compasses as follows:

2. Draw a circle with A as a center and AB as a radius. All radii of this circle are equal to AB.

3. Then, draw a circle with B as a center and BA as a radius. All radii of this circle are equal to AB.

4. From point C, where the circles intersect, draw lines CA and CB.

5. Triangle ABC is the required equilateral triangle.

4. Similarly, draw an isosceles triangle, given its base AB and one of its equal sides AC.

5. Similarly, draw a scalene triangle, given its three sides.

PROOFS IN GEOMETRY

76. Need of Proofs. It is not sufficient in geometry to draw accurate figures. We must *prove* that the figures are what we take them to be. That the appearance of figures may be misleading is seen from a few examples.

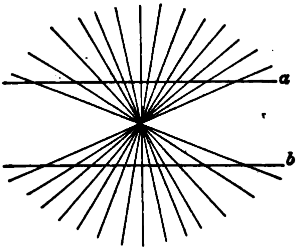


FIG. 1

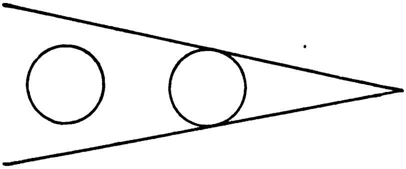


FIG. 2

1. In Fig. 1 the lines *a* and *b* appear curved. Test them with a ruler. The figure misleads.

2. In Fig. 2 which appears to be the larger circle? Test by measurement.

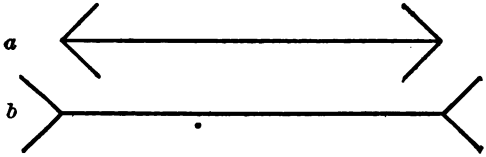


FIG. 3

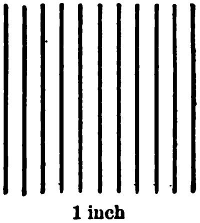


FIG. 4

3. In Fig. 3 which appears to be the longer, *a* or *b*?

4. In Fig. 4 compare its width with its height and test.

77. Geometric Proof. In geometry we prove a statement by giving reasons sufficient to show that the statement is true. These reasons may be statements that have previously been proved, or they may be statements whose truth is accepted without proof.

78. Axiom. A general statement accepted without proof is called an **axiom.**

Many axioms have already been used in arithmetic and algebra, sometimes with formal statements, more frequently without. The following may be mentioned here:

I. *If two numbers are equal to a third number, they are equal to each other.*

A special consequence of this axiom is: A number may always be substituted for its equal.

II. *If equal numbers be added to equal numbers, the results are equal numbers.*

III. *If equal numbers be subtracted from equal numbers, the results are equal numbers.*

IV. *If equal numbers be multiplied by equal numbers, the results are equal numbers.*

V. *If equal numbers be divided by equal numbers (not zero), the results are equal numbers.*

VI. *The same integral powers, or the same positive roots of equal numbers, are equal.*

The statement as to powers is already covered by IV, but the form of statement of VI is sometimes convenient.

Other axioms less frequently met in arithmetic and algebra are:

VII. *The whole is greater than any of its parts and is equal to the total of all of its parts.*

VIII. *If of three quantities the first is greater than the second, and the second is greater than the third, then the first is greater than the third.*

IX. *If the members of an inequality be increased by, diminished by, multiplied by, or divided by the same positive number, the result will be an inequality in the same sense.*

Thus, if $a > b$, then $a + x > b + x$, $a - x > b - x$, $ax > bx$, and $\frac{a}{x} > \frac{b}{x}$.

X. *The sum of the larger members of two or more inequalities is larger than the sum of the smaller members.*

Thus, if $a > b$ and $p > q$, then $a + p > b + q$.

79. Relation of the Above Axioms to Geometry. We have defined, both for straight line segments and for angles, what is meant by equality, addition and subtraction. Axioms I, II and III are evidently true for segments and angles as well as for numbers, but as we have not defined how to multiply or divide a segment by a segment, or an angle by an angle, Axioms IV and V have yet no meaning with respect to these magnitudes.

Axiom IV has, however, a meaning if the multiplicand is a segment, or an angle, and the multiplier is a positive integer, for such multiplication is simply repeated addition.

Thus, 3 times segment AB is $AB + AB + AB$.

In this case we have:

XI. *If equal segments, or angles, be multiplied by equal positive integers, the results are equal segments or angles.*

Postulate V, given below, covers the case of division of segments and angles by positive integers, and, since we multiply by a fraction by multiplying by its numerator and dividing the result by its denominator, it follows that:

XII. *If equal segments, or angles, be multiplied by equal positive fractions, the results are equal segments or angles.*

Since we divide by a fraction by multiplying by its reciprocal, the corresponding statement for fractional divisors also holds true in consequence of Postulate V.

80. Postulates. The above axioms are assumptions of arithmetic and algebra that hold also for certain magnitudes of geometry. There are other assumptions that relate only to geometry, and these may be called **postulates**.

The modern tendency is to employ the term "postulate" for all the assumptions of geometry. The term has sometimes been limited to assumptions governing constructions only.

The following postulates may be mentioned:

I. *Through two distinct points, one and only one straight line can be drawn.*

This is another way of saying that two points determine a straight line. It follows, also, from this postulate, that two distinct straight lines can have at most one point in common.

II. *A line segment may be produced, or extended, to any required length.*

III. *Through a point an unlimited number of straight lines may be drawn.*

IV. *The straight line is the shortest line that can be drawn between two points.*

It follows from this postulate that the sum of any two sides a, b, c, of a triangle is greater than the third side, and their difference is less than the third side.

For $a + b > c$ by this postulate; hence by subtracting b from both members of the inequality $a > c - b$.

V. *A segment may be thought of as divided into any number of equal segments.*

As a result of Postulate V, if two equal segments are each divided into the same number of equal segments, any one of the parts of one is equal to any one of the parts of the other.

VI. *A circle may be drawn with any given point as the center and any given line segment as the radius.*

VII. *Any figure may be moved from one place to another without changing its shape or size.*

VIII. *Through a given point on a line there passes one and only one line perpendicular to the given line.* Sec. 45.

81. Theorem. A geometric statement to be proved is called a **theorem.**

82. Problem. A statement requiring a geometric construction is called a **problem.**

83. Proposition. A **proposition** is either a theorem or a problem.

84. Corollary. A statement whose truth can be established readily in connection with the proof of a proposition, is called a **corollary** to the proposition.

HISTORICAL NOTE

THALES

The beginnings of geometry arose from the needs of practical life. To construct the great Pyramids of Egypt and to preserve the boundaries of the ever-shifting valley of the Nile required the invention and use of instruments based upon geometric principles. It was from the practical Egyptians that the Greeks in the seventh century B.C. learned enough of elementary geometry to arouse their curiosity and to give a basis for speculative thinking. The nations that flourished prior to the time of the Greeks developed many arts, but the Greeks led in science and philosophy. Among the Greeks who visited Egypt as a matter of commercial enterprise was Thales of Miletus, who lived about 600 B.C. So impressed was he with the accomplishments of the Egyptians that he remained among them some time to acquire the learning of the priests. He subsequently returned to Greece, introduced the new science of geometry, founded the Ionic school of philosophy, and attained the honor of being counted one of "the seven wise men."

Thales of Miletus

Thales, when in Egypt, measured the heights of the Pyramids by comparing their shadows with the shadow of a staff, and measured the distance of ships at sea by applying properties of triangles.

BOOK I

ANGLES, TRIANGLES AND QUADRILATERALS

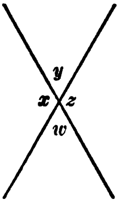
85. Preparatory.

1. What is the sum of angles x and y? Of angles y and z?

2. If $\angle x$ is 130°, how many degrees in $\angle y$? In $\angle z$? In $\angle w$?

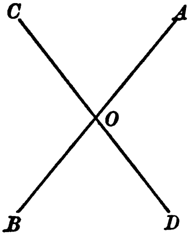
3. Compare the values of x and z; of y and w.

4. Answer questions 2 and 3 when $\angle x = 112°$.



PROPOSITION I. THEOREM

86. *The vertical angles formed by two intersecting straight lines are equal.*



Given two intersecting straight lines, AB and CD, making vertical $\measuredangle s$ AOC and BOD.

To prove that $\angle AOC = \angle BOD$.

STATEMENTS	REASONS
Proof. 1. $\angle AOC + \angle DOA =$ a straight angle.	1. Line CD is given as a straight line; hence the sum of $\angle AOC$ and $\angle DOA$ is a straight angle. Secs. 38, 42
2. And $\angle BOD + DOA =$ a straight angle.	2. AB is given as a straight line. Secs. 38, 42
3. Therefore, $\angle AOC + \angle DOA = \angle BOD + \angle DOA$.	3. Quantities equal to the same quantity are equal to each other. Sec. 78, I
4. Then, $\angle AOC = \angle BOD$.	4. Subtract $\angle DOA$ from both members of step 3. Sec. 78, III

Q.E.D.

Similarly, show that $\angle BOC = \angle DOA$.

NOTE. — The form used above will usually be employed in what follows. The *statements* constituting the proof will appear on the left, the *authority* or reason for each statement opposite it on the right. When the reason is easily seen, it may be given only in part or by reference, or not at all. In such cases the pupil should always complete it.

87. The assumptions made in the proposition, that is, the statements given as facts, are often called the **hypothesis**, and the statement to be proved is called the **conclusion**. The argument that leads from the hypothesis to the conclusion constitutes the **proof**, or **demonstration**, of the proposition.

88. To prove a proposition:

First. Note exactly what is given in the statement of the proposition.

Second. Note exactly what is to be proved.

Third. Indicate the conditions in a suitable figure.

Fourth. Find the previous results (assumptions or propositions) which will establish the truth of the new proposition.

Fifth. Scrutinize the result to make sure that you have actually proved that which was to be proved. Only after this has been found to be the case, is the traditional Q.E.D. in order.

EXERCISES

1. How many angles are there in Fig. 1, none of which overlaps any other? Name the pairs of equal angles among these angles.

2. How many degrees in the sum of ∠s x, y, z? Of ∠s y, z, w? Of ∠s $y, x, 30°$?

3. In Fig. 1 find the number of degrees in ∠s x, y, z and w.

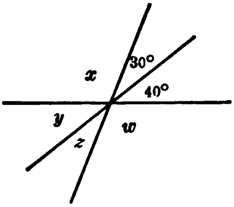


FIG. 1

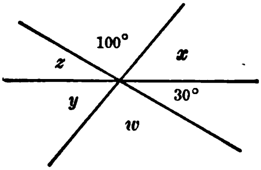


FIG. 2

4. Do the same thing for Fig. 2.

5. In Fig. 3 find the number of degrees in ∠s x, y and z.

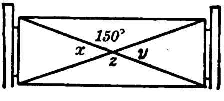


FIG. 3

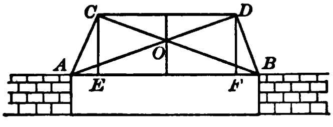


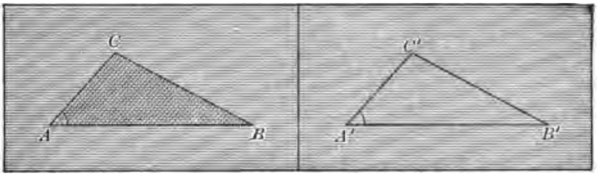
FIG. 4

6. In Fig. 4 if $\angle AOB = 140°$, find the number of degrees in ∠s BOD, DOC and COA.

7. Two intersecting lines form an angle of 30°. How many degrees in each of the other angles? If the given angle is 65°, how many degrees in each of the other angles? If the given angle is 5° 45′, answer the same question.

89. Preparatory.

Cut out a paper triangle, ABC, in the figure. Place the paper triangle on another sheet of paper and trace along two sides, AB and AC. Remove the paper triangle. You have thus traced two sides and their included angle belonging to the given triangle ABC. Draw a straight line connecting the end points, B' and C', of the two sides. A second triangle, $A'B'C'$, is thus formed, having two of its sides, $A'B'$ and $A'C'$, and their included angle, $B'A'C'$, equal to the sides AB and AC and the angle BAC of the first triangle. Can the two triangles be made to coincide completely?



90. Superposition. When one figure is placed upon another so as to coincide with it, the process is called **superposition,** and the figure so placed is said to be **superposed** upon the other.

91. Equal or Congruent Figures. We have already defined line segments and angles as equal, when they can be made to coincide. So generally, any two geometric figures that can be made to coincide throughout are said to be **equal,** or **congruent.**

Symbol for Equality or Congruence. The sign of equality ($=$), used in arithmetic or algebra, may also be used to denote geometric equality or congruence.

Other symbols for congruence, such as $\cong$, $\equiv$, have been proposed, but none of them has found general acceptance.

PROPOSITION II. THEOREM

92. *If two triangles have two sides and the included angle of the one equal respectively to two sides and the included angle of the other, the triangles are congruent.*

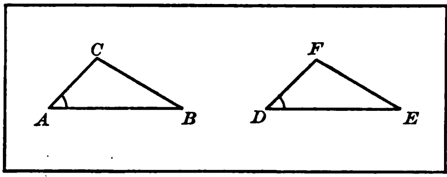


FIG. 1

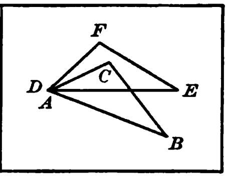


FIG. 2

Given $\triangle ABC$ and $\triangle DEF$ in which $AB = DE$, $AC = DF$ and $\angle BAC = \angle EDF$.

To prove that $\triangle ABC$ is congruent to $\triangle DEF$.

Proof. 1. $\triangle ABC$ can be placed on $\triangle DEF$ so that A will fall on D. Fig. 2.	1. A figure can be moved from one place to another without changing its size and shape. Sec. 80, VII
2. Then, $\triangle ABC$ can be turned about D as a pivot until AB coincides with DE.	2. As in (1), also segment AB is given equal to segment DE.
3. AC will lie along DF.	3. $\angle BAC$ is given equal to $\angle EDF$. Sec. 36
4. And C will fall on F.	4. AC is given equal to DF. Sec. 22
5. Finally, CB will lie on FE.	5. These lines have two points in common. Sec. 80, I
6. Therefore, $\triangle ABC$ is congrugent to $\triangle DEF$.	6. These two figures coincide. Sec. 91

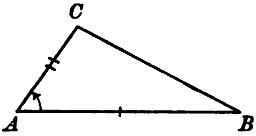
Q.E.D.

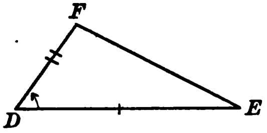
93. COROLLARY. *If two right triangles have the sides including the right angles respectively equal, the triangles are congruent.*

94. Homologous Parts. When congruent figures are superposed, any selected point of one will coincide with a definite point of the other. The points of such a pair are called **corresponding** or **homologous** points. Similarly, lines, angles and other parts that coincide when the figures are superposed are called **corresponding** or **homologous** parts.

In proofs, it is often helpful to designate the homologous parts by some system of marks, or by colors.

For example, the figures in the above theorem might have been marked as follows:

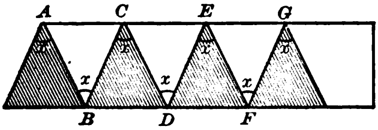




The homologous sides AB and DE each with a single mark, the homologous sides AC and DF each with a double mark, and the homologous angles BAC and EDF with an arrow.

EXERCISES

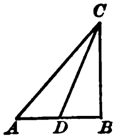
1. In this diagram of a tile border, the sides AB, CD and EF are all equal, and BC, DE and FG are all equal. The angles marked x are also equal. Name three congruent triangles. Why are they congruent?

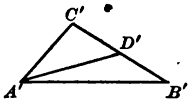


2. In the figure, given AC to be 2 in., what is the length of BD? Of DF? Of AE? Of CG? Of AG? Give the reasons for your answers.

3. Suppose that $\angle BAC = 60°$, what is the size of $\angle CDB$? $\angle EFD$? $\angle DCE$?

95. Median. The line connecting any vertex of a triangle with the mid-point of the opposite side is called the **median** of the triangle to that side.

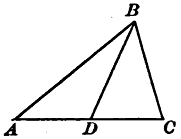


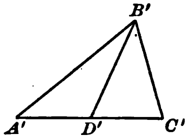


Thus, in $\triangle ABC$ the line CD is the median to the side AB, and in $\triangle A'B'C'$ the line $A'D'$ is median to the side $C'B'$.

EXERCISES

1. $\triangle ABC$ and $\triangle A'B'C'$ are congruent. Show by superposition that the medians BD and $B'D'$ are equal.



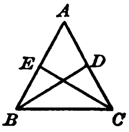


2. Draw two triangles like those in Exercise 1, and draw their medians to the sides BC and $B'C'$. Are these medians equal? Why?

3. In the figures of Exercise 2, draw the medians to the sides BA and $B'A'$. Are they equal? Why?

4. Draw two congruent triangles ABC and $A'B'C'$ and assume the bisectors of $\angle$s A and A'; let them cut the sides BC and $B'C'$ in D and D'. Are AD and $A'D'$ equal? Why?

5. In this isosceles triangle, CE and BD are medians.



Prove that $\triangle BDA$ is congruent to $\triangle CEA$.

How do the medians CE and BD compare? Why?

6. How many equal medians has an equilateral triangle? Why?

Answer each question and give the reason:

7. In Fig. 1, $AC = AD$. State the length of BD.

8. Compare $\angle ACB$ with $\angle ADB$.

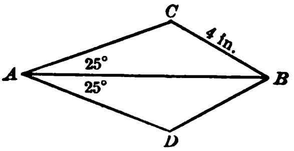


FIG. 1

9. In Fig. 2, given that the angles marked a are equal and that $AB = AC = AD = AE$, compare the lengths BC, CD, DE.

10. Name an angle equal to $\angle ADE$; to $\angle ACB$.

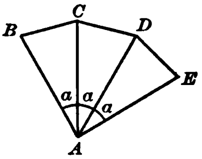


FIG. 2

11. What kind of triangles are those in Fig. 2?

12. Given that the perimeter of Fig. 1 above is 22 in. Find AC.

13. In Fig. 2, given that $AB = 10$ in., $BC = 4$ in. Find the sum of the lengths of all the straight lines in the figure.

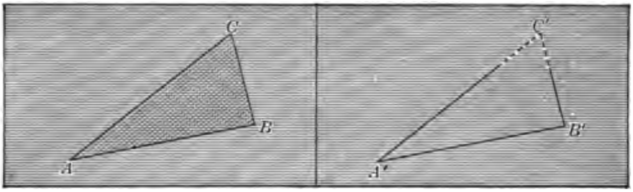
14. In the figure of Ex. 5, given that $BC = 11$ in., $BE = 6$ in., $CE = 8$ in., find the sum of the lengths of all the lines in the figure.

15. In a figure like Fig. 2, given that BC is half as long as BA and letting x represent the length BC, what represents that of BA? Of the sum of all the straight lines in the figure? Given that the sum of these lines is 66 in., find the length of BA.

16. In the figure of Ex. 5, given that BA is 2 in. longer than BD, and that the sum of their lengths is 32 in. Find BA. Find DA and BD, and the perimeter of BDA.

17. In a figure like that of Ex. 5, given that BD is 3 in. longer than AD. Letting x represent the length of AD, what represents the perimeter of BDA? If this perimeter is 24 in., find AD.

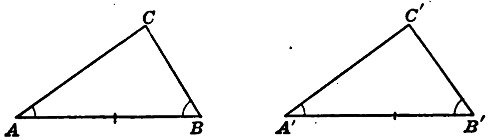
96. Preparatory. Make a paper triangle like ABC. Place the paper triangle on another sheet of paper and trace along one side as AB. Then trace a little distance along each of the other sides from the end points of the first side. Remove the triangle.



You have thus traced two angles and the included side of the given triangle, ABC. Prolong the incomplete sides until they meet. A second triangle, $A'B'C'$, is thus formed, having two of its angles and their included side equal to the corresponding parts of the first triangle. Can the two triangles be made to coincide completely?

PROPOSITION III. THEOREM

97. *If two angles and the included side of one triangle are equal respectively to two angles and the included side of another triangle, the triangles are congruent.*



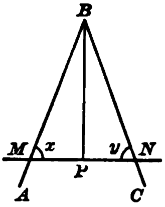
Given △ ABC and △ $A'B'C'$, having $AB = A'B'$, $\angle A = \angle A'$ and $\angle B = \angle B'$.

To prove that △ ABC is congruent to △ $A'B'C'$.

Proof. 1. Place $\triangle ABC$ on $\triangle A'B'C'$ so that AB coincides with $A'B'$.	1. A figure can be moved from one place to another without change of shape and size. Sec. 80, VII
2. Side AC will then lie along $A'C'$ and BC will lie along $B'C'$.	2. When two equal angles are superposed, the sides lie on each other. Sec. 36
3. Then, C will fall at C'.	3. The line AC lying on $A'C'$, and the line BC lying on $B'C'$, must intersect in a point as C', because two lines can intersect in only one point. Secs. 17, 18
4. Therefore, $\triangle ABC$ is congruent to $\triangle A'B'C'$.	4. The two triangles coincide throughout. Sec. 91

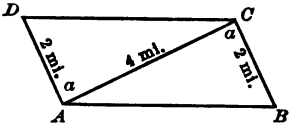
Q.E.D.

EXERCISES

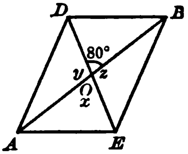


1. In the figure, BP is the bisector of $\angle ABC$, and MN is perpendicular to BP. Prove that MN makes equal angles with the sides AB and BC, and also that point P bisects the segment MN.

2. In the figure, the angles marked a being equal, and the distances being as indicated, compare the distance between A and B with that between C and D. Name two pairs of equal angles.



3. How many degrees in $\angle y$? In $\angle z$? In $\angle x$? If O is the middle point of AB, and also of DE, name a pair of congruent triangles. Hence, name a pair of equal lines; two pairs of equal angles. Name a second pair of congruent triangles, and hence another pair of equal lines, and two pairs of equal angles.

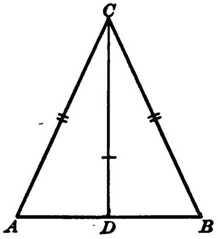


98. Auxiliary Lines. Sometimes an additional line or lines drawn in a figure will reveal properties that are connected with previous propositions, and thus lead to a proof; such lines are called **auxiliary lines.**

No rule can be given for drawing auxiliary lines. It is often necessary to try several such lines before a method of proof is established. We consider previous propositions, axioms and definitions, and assume such lines as we think will connect the new proposition with what we have proved before. It is permissible to assume any lines that we know *exist*, even though we may not know a method for actually drawing them with instruments. Thus, we know every angle must have a bisector; we may accordingly draw an auxiliary line to *represent* that bisector, even before we have learned how to construct a bisector with ruler and compasses.

PROPOSITION IV. THEOREM

99. *In an isosceles triangle, the angles opposite the equal sides are equal.*



Given the isosceles $\triangle ABC$, with $CA = CB$.

To prove that $\angle A = \angle B$.

Proof. 1. Suppose the line CD drawn so as to bisect $\angle ACB$. Sec. 98

2. $\angle ACD = \angle DCB$. Sec. 37

3. $\triangle$ ACD and BCD are congruent. Sec. 92

4. Hence, $\angle A = \angle$ B. Homologous angles. Sec. 94. Q.E.D.

NOTE.—It is to be permanently understood that expressions like "draw CD" or "join C and D" or "the line connecting C and D" mean to do so by a straight line, unless otherwise specified.

100. Cor. 1. *In an isosceles triangle, the bisector of the angle between the equal sides bisects the base.*

For, in the above figure, $\triangle ADC$ is congruent to $\triangle DBC$. Therefore, $AD = BD$, being homologous sides. Sec. 94.

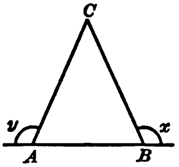
101. Cor. 2. *In an isosceles triangle, the bisector of the angle between the equal sides is perpendicular to the base.*

For the two angles at D are equal (Why?). Hence, each is half of a straight angle. Sec. 46.

102. Cor. 3. *An equilateral triangle is also equiangular.*

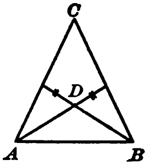
Note.—It is to be understood permanently that full proofs of all corollaries are to be given by the pupil, and proofs of theorems and corollaries to be completed whenever they are given in outline only.

EXERCISES

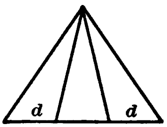


1. In the figure, $\triangle ABC$ is isosceles. Prove that $\angle x = \angle y$.

2. In isosceles $\triangle ABC$, AD and BD bisect $\angle$s A and B. Prove that $DA = BD$.



3. If A and B are points one on each arm of an $\angle ACB$ and equidistant from the vertex C, then the bisector of angle ACB is the perpendicular bisector of the line AB.

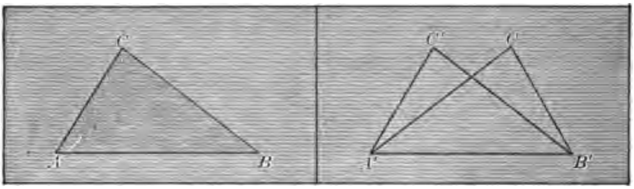


4. If the base of an isosceles triangle is produced the same distance d in both directions, and the end points joined to the opposite vertex, prove that the two triangles thus formed outside of the given triangle are equal.

5. The base of an isosceles triangle is divided into three equal parts by the points P and Q. Prove that the lines drawn to P and Q from the opposite vertex are equal.

103. Preparatory.

1. Cut a paper triangle like ABC in the figure.
2. Then, trace it, making a duplicate like $A'B'C'$.
3. Turn $\triangle ABC$ over on its duplicate. It will take a position like $A'B'C$.



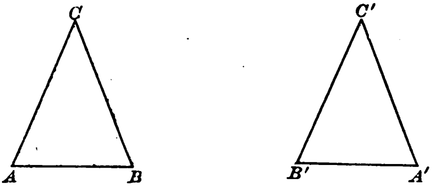
4. At B' find two angles of $\triangle ABC$. Do the same at A'.
5. If $\angle A$ were equal to $\angle B$, where would line $B'C$ fall? Where would line $A'C$ fall?

PROPOSITION V. THEOREM

104. *If two angles of a triangle are equal, the sides opposite these angles are equal and the triangle is isosceles.*

Given $\triangle ABC$ having $\angle A = \angle B$.

To prove that $AC = BC$.



Proof. 1. Suppose ABC to be turned over and let it assume the position $B'A'C'$. Then, $\angle B' = \angle B$. Sec. 80, VII

2. But $\angle A = \angle B$. Given

3. Hence, $\angle A = \angle B'$. From (1) and (2). Sec. 78, I

4. Similarly, $\angle B = \angle A'$. As in step 1

5. $B'A' = AB$. Given

6. Therefore, △ ABC and $B'A'C'$ are congruent as they stand. Sec. 97

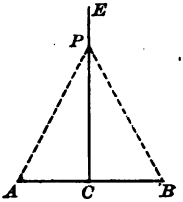
7. Hence, $BC = A'C'$. Homologous sides. Sec. 94

8. But $AC = A'C'$. As in step 1

9. Hence, $BC = AC$ and △ ABC is isosceles. Sec. 78, I; Sec. 63. Q.E.D.

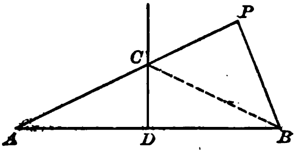
105. Cor. 1. *If the three angles of a triangle are equal, the triangle is equilateral.*

106. Cor. 2. *Every point on the perpendicular bisector of a line segment is equidistant from the ends of the segment.*



Suggestion. Use Sec. 92 to prove △ APC and CPB congruent.

On the other hand, every point not on the perpendicular bisector of a line segment is unequally distant from the end points of the segment.



Suggestion. Assume such a point P at the right of the perpendicular. Let AP cut the perpendicular at C. Connect C with B. Prove that $PA = PC + CB$; but this is greater than PB (Why?). Hence, line $PA > PB$.

EXERCISES

1. Show that any point of the bisector of the angle between the equal sides of an isosceles triangle is equally distant from the extremities of the base.

2. Prove that a scalene triangle has no equal angles.

3. Draw a square and one of its diagonals. Show that the four acute angles in the resulting figure are all equal.

4. In the upper figure of Corollary 2, Sec. 106, prove that $\angle A = \angle B$.

PROPOSITION VI. THEOREM

107. *If in two triangles the three sides of the one are equal to the corresponding sides of the other, the triangles are congruent.*

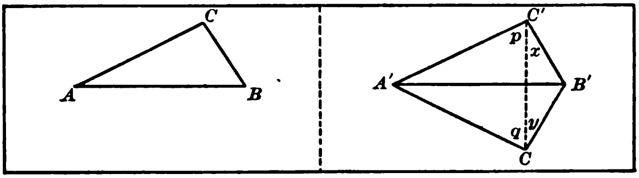


FIG. 1 FIG. 2

Given $\triangle$s ABC and $A'B'C'$ with $AB = A'B'$, $AC = A'C'$ and $BC = B'C'$.

To prove that $\triangle ABC$ is congruent to $\triangle A'B'C'$.

Proof. 1. Place $\triangle A'B'C'$ next to $\triangle ABC$ so that AB coincides with $A'B'$ as in the figure. Sec. 80, VII

2. Draw the auxiliary line CC'. Sec. 98
3. $\triangle C'B'C$ is isosceles for $C'B' = B'C$. Given
4. $\triangle C'A'C$ is isosceles for $A'C' = A'C$. Given
5. $\angle x = \angle y$ and $\angle p = \angle q$. Sec. 99
6. $\angle C = \angle C'$. Secs. 78, II; 79
7. $\therefore \triangle ABC$ is congruent to $\triangle A'B'C'$. Sec. 92. Q.E.D.

EXERCISES

1. In Fig. 1, given $AD = BC$, $AC = BD$; prove that $\triangle ABC$ is congruent to $\triangle ABD$.

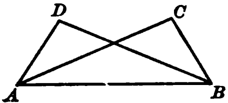


FIG. 1

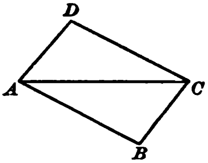


FIG. 2

2. In Fig. 2, given $AB = CD$, $AD = BC$; prove $\angle B = \angle D$.

3. If upon the sides of an angle any two points be taken equally distant from the vertex, any point equally distant from these points is a point either of the bisector of the angle, or of the bisector produced.

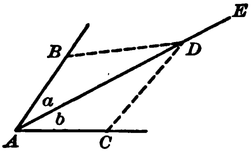


FIG. 1

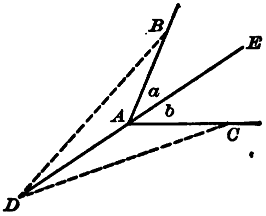
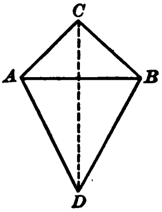


FIG. 2

Suggestion. In both figures, given $AB = AC$, and $BD = DC$. To prove that $\angle a = \angle b$, show that $\triangle ABD$ and $\triangle ACD$ are congruent. In Fig. 2, the line DA is the extension of the bisector AE.

4. ABC and ABD are two isosceles triangles on the same base AB, but on different sides of it. (1) Prove $\angle CAD = \angle CBD$.



(2) Connect C and D and prove that $\triangle$s ACD and CBD are congruent.

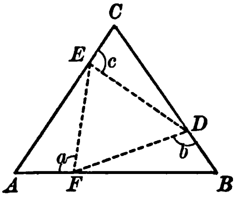
(3) How does CD divide $\angle C$? Also $\angle D$?

(4) Suppose these triangles are brought to coincide, show that CD bisects AB and is perpendicular to it.

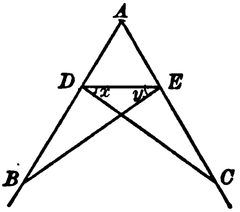
5. Prove that the triangle formed by joining the mid-points of the sides of an equilateral triangle is also equilateral.

6. Prove that the two triangles formed by joining the mid-point of the base of an isosceles triangle to the mid-points of the sides are congruent.

7. If $\triangle ABC$ is equilateral, and if $AF = BD = CE$, prove that $\triangle$s AFE, FBD and DCE are congruent. What kind of triangle is DEF?

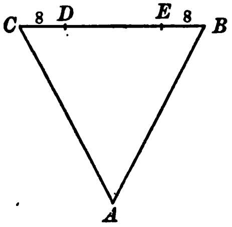


8. Given $AD = AE$, $DB = EC$, prove $\angle x = \angle y$.



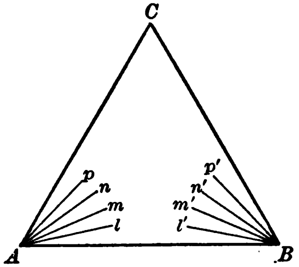
9. From the point in which the bisectors of the base angles of an isosceles triangle intersect, a line is drawn to the third vertex of the triangle. Prove that the line bisects the angle at the vertex.

10. A playground has the shape of an isosceles triangle, the equal sides meeting at A. Points D and E are 8 ft. from B and C respectively on BC. Show that D and E are equally distant from A.



11. Two triangles are congruent if two sides of one are equal to two sides of the other and if the median to one of these sides is equal to the median to the corresponding side of the other triangle.

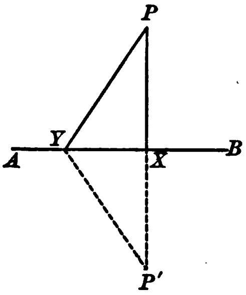
12. Given that $AC = BC$, and that the lines l, m, and so on, and l', m', and so on, divide the $\angle$'s A and B into five equal angles. Prove that the four pairs of corresponding lines taken in order from AB, l and l', m and m', etc., all intersect on the perpendicular bisector of AB.



13. Prove the preceding statement, if the number of divisions of each angle is n.

PROPOSITION VII. THEOREM

108. *From a point without a straight line, only one perpendicular can be drawn to the line.*



Given $PX \perp AB$, and Y, a point of AB other than X.

To prove that PY cannot be $\perp AB$.

Proof. 1. Assume PX produced to P' so that $PX = P'X$. Connect P' and Y. Sec. 98

2. $PY = P'Y$. Sec. 106

3. Therefore, $\triangle$s PXY and $P'XY$ are congruent. Sec. 107

4. Therefore, $\angle PYX = \angle P'YX$. Sec. 94

5. Hence, $\angle PYX = \frac{1}{2} \angle PYP'$.

6. But PYP' is not a straight line. Hence, PYP' is not a straight angle. Secs. 17, 38

7. Therefore, $\angle PYX$ is not a right angle. Step (5) and Sec. 46

8. Therefore, PY is not perpendicular to AB. Sec. 47

9. Since Y was taken as any point of AB other than X, and PY is not perpendicular to AB, PX is the only perpendicular from P. Q.E.D.

PROPOSITION VIII. THEOREM

109. *If two right triangles have the hypotenuse and an acute angle of the one respectively equal to the hypotenuse and an acute angle of the other, the triangles are congruent.*

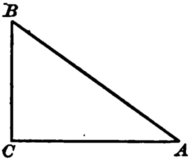


FIG. 1

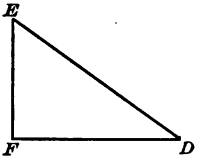


FIG 2

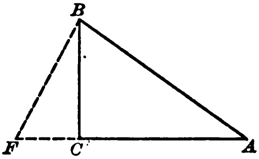


FIG. 3

Given triangles ABC and DEF, in which $\angle C = \angle F = 90^\circ$, $\angle A = \angle D$ and $AB = ED$.

To prove that $\triangle$s ABC and DEF are congruent.

Proof. 1. Place $\triangle DEF$ upon $\triangle ABC$ so that ED coincides with AB, and $\angle D$ with $\angle A$. Sec. 80, VII

2. Then, the vertex F will fall somewhere upon the side AC, as at F in Fig. 3.

3. It cannot fall at a point other than C, for there would be two perpendiculars from B to the line AC, contrary to Sec. 108.

4. Hence, EF coincides with BC, and $\triangle$s ABC and DEF are congruent. Q.E.D. Sec. 91

110. COROLLARY. *The perpendicular from the apex of an isosceles triangle to the base bisects the base and the angle at the apex.*

For the two triangles into which the perpendicular divides the isosceles triangle are congruent by the theorem.

EXERCISES

1. Perpendiculars drawn to the sides of an isosceles triangle from the midpoint of the base are equal.

2. If two points are on opposite sides of a line and are equidistant from it, the line joining the points is bisected by the given line.

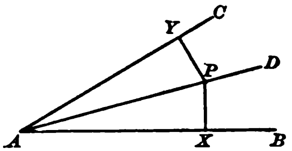
3. The perpendiculars erected to the base of an isosceles triangle, at points equidistant from its midpoint and terminated by the sides of the triangle, are equal.

4. If a perpendicular to a given line is drawn and obliques from a point of the perpendicular to the given line make equal angles with the perpendicular, the obliques are equal.

PROPOSITION IX. THEOREM

111. *Every point of the bisector of an angle is equidistant from the sides of the angle, and every other point within the angle is unequally distant from its sides.*

I. To prove that every point of the bisector is equally distant from the sides of the angle.



Given $\angle BAC$, AD its bisector and P any point of AD.

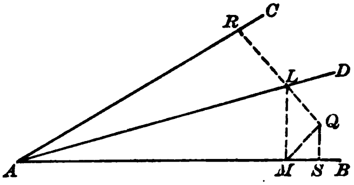
To prove that P is equidistant from AC and AB.

Proof. 1. Assume $PX \perp AB$ and $PY \perp AC$. Sec. 98

2. Then, $\triangle AXP$ is congruent to $\triangle APY$. Sec. 109

3. $\therefore PX = PY$. Sec. 94. Q.E.D.

II. Any point within the angle and not on the bisector is unequally distant from the sides of the angle.



Given a point Q not on the bisector AD and between AD and AB. Assume $QR \perp AC$, and $QS \perp AB$.

To prove that $QR > QS$.

Proof. 1. Let QR meet AD in L. Assume $LM \perp AB$ and connect M with Q. Sec. 98

2. $QR = QL + LM$. Sec. 111, I

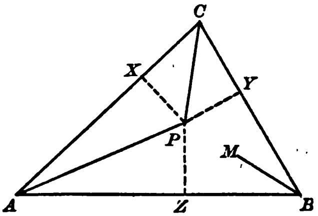
3. $QL + LM > QM$, which is greater than QS. Why?

4. $\therefore QR > QS$. From (2) and (3) Sec. 78, VIII. Q.E.D.

EXERCISES

1. In the figure, assume PC to bisect $\angle C$, and PA to bisect $\angle A$:

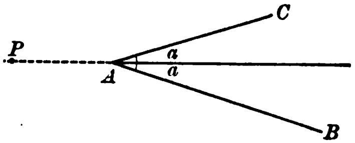
(1) Show that $PX = PY = PZ$.



(2) If BM bisects $\angle B$, show that it must pass through P.

2. In Ex. 1, if $\triangle ABC$ is isosceles and AB is the base, prove that CPZ is a straight line.

3. In the figure, prove that P is equidistant from the extensions of AB and AC.



4. In the above figure, compare angle PAC and angle PAB.

PROPOSITION X. THEOREM

112. *If two right triangles have the hypotenuse and a side of the one respectively equal to the hypotenuse and a side of the other, the triangles are congruent.*

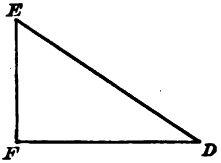


FIG. 1

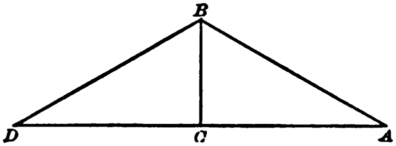


FIG. 2

Given the triangles ABC and DEF, with right angles at C and F. Also $AB = ED$, and $BC = EF$.

To prove that $\triangle$s ABC and DEF are congruent.

Proof. 1. Place the triangles as in Fig. 2, so that the two right angles shall be adjacent, and so that the equal sides BC and EF coincide. Sec. 80, VII

2. Then, DCA is a straight line. Secs. 38, 46

3. And $AB = BD$. Given

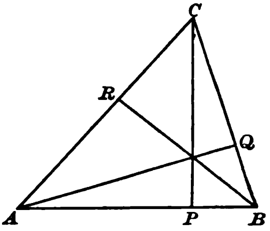
4. Therefore, $\triangle ABD$ is isosceles.

5. BC bisects the base AD. Sec. 110

6. Hence, $\triangle ACB$ is congruent to $\triangle BCD$ or $\triangle DEF$. Sec. 107. Q.E.D.

113. Corollary. *If two oblique lines are drawn from a point P to a straight line so as to make equal acute angles with it, the oblique lines are equal.*

114. Altitudes of Triangles. The perpendicular distance from any vertex of a triangle to the side opposite is called the *altitude* of the triangle from that vertex, and on that side.



Thus, CP is the altitude of $\triangle ABC$ from the vertex C to the base AB. Similarly, AQ is the altitude from A to CB, and BR is the altitude to AC.

EXERCISES

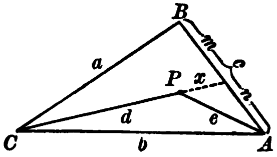
1. The altitudes on the equal sides of an isosceles triangle are equal.

2. If two altitudes of a triangle are equal, the triangle is isosceles.

3. If the three altitudes of a triangle are equal, the triangle is equilateral.

Proposition XI. Theorem

115. *If two straight lines are drawn from a point within a triangle to the end points of a side, their sum is less than the sum of the other two sides of the triangle.*



Given $\triangle ABC$ and point P within the triangle.

To prove that $CP + PA < BC + BA$,

or $d + e < a + c$.

Proof. 1. $d + x < a + m$. Sec. 25, 1

2. $e < n + x$. Why?

3. $d + e + x < a + m + n + x$. Sec. 78, X

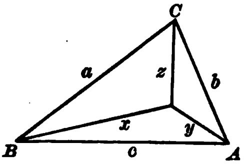
4. $\therefore d + e < a + m + n$, or $d + e < a + c$. Sec. 78, IX. Q.E.D.

EXERCISES

1. The sum of the lines drawn from any point within a triangle to vertices of the triangle is less than the perimeter of the triangle.

Suggestions.

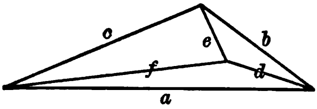
(1) $x + z < b + c$.
(2) $y + z < a + c$.
(3) $x + y < a + b$.
(4) Add and divide by 2.



2. Show that the sum of the lines joining any point within a triangle to the three vertices is greater than half the sum of the sides of the triangle.

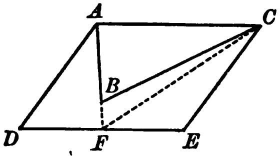
Suggestions.

(1) $e + d > b$.
(2) $d + f > a$.
(3) $e + f > c$.
(4) Add and divide by 2.



3. Show that in the figure:

$$AD + DE + EC > AF + FE + EC > AF + FC > AB + BC.$$



116. The *distance from a point to a line* is the length of the perpendicular from the point to the line.

PROPOSITION XII. THEOREM

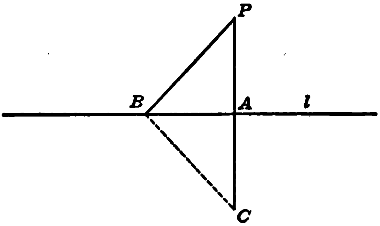
117. *If from a point without a straight line, lines are drawn to points of the given line:*

(I) *The perpendicular is the shortest line that can be drawn.*

(II) *Two oblique lines meeting the given line at equal distances from the foot of the perpendicular are equal.*

(III) *Two oblique lines meeting the given line at unequal distances from the foot of the perpendicular are unequal, and that one is the longer that meets the given line at the greater distance from the foot of the perpendicular.*

To prove I:



1. Let P be the given point and PA be perpendicular to l. Then, let PB be any other line from P to l.

2. Produce PA to C so that $PA = AC$, and connect B and C.

3. Then, $PB = BC$. Sec. 106

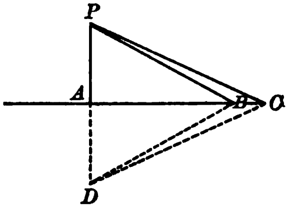
4. But, $PB + BC > PA + AC$, or $2\,PB > 2\,PA$. Sec. 25, I

5. Therefore, $PB > PA$, that is, the perpendicular is shorter than any oblique line from P to l. Sec. 78, IX

To prove II:

Draw the figure and apply Proposition II, Sec. 92.

To prove III:



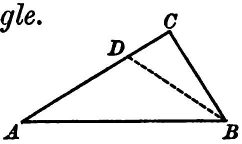
Given $AC > AB$.

To prove $PC > PB$.

1. Extend PA to D, making $AD = PA$. Sec. 98
2. Then, $PC + CD > PB + BD$. Sec. 115
3. But, $CD = PC$, and $BD = PB$. For AC is the perpendicular bisector of PD. Sec. 117, II
4. $\therefore 2\,PC > 2\,PB$, or $PC > PB$. By steps (2) and (3). Sec. 78, I, IX. Q.E.D.

PROPOSITION XIII. THEOREM

118. *If two angles of a triangle are unequal, the side opposite the larger angle is longer than the side opposite the smaller angle.*



Given $\triangle ABC$ with $\angle B > \angle A$.

To prove that $AC > BC$.

Proof. 1. Let BD be drawn so that $\angle ABD = \angle A$. Sec. 98

2. Then, $\triangle ABD$ is isosceles, and $AD = DB$. Why?

3. $DB + DC = AD + DC$, or AC.

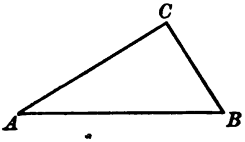
Adding DC to both members in step 2, to obtain AC as one member.

4. But, $DB + DC > BC$. Why?

5. Therefore, $AC > BC$. From step (3). Sec. 78, I. Q.E.D.

PROPOSITION XIV. THEOREM

119. *If two sides of a triangle are unequal, the angle opposite the longer side is larger than the angle opposite the shorter side.*



Given $\triangle ABC$ with $AC > BC$.

To prove that $\angle B > \angle A$.

Proof. 1. If $\angle B$ were equal to $\angle A$, then AC would be equal to BC. Sec. 104

2. But AC is not equal to BC, being greater than it by hypothesis. Hence, $\angle B$ is not equal to $\angle A$.

3. If $\angle B$ were less than $\angle A$, then AC would be less than BC. Sec. 118

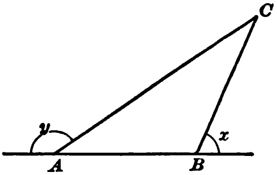
4. But AC is not less than BC. Hence, $\angle B$ is not less than $\angle A$.

5. Therefore, $\angle B > \angle A$.

Because we have shown that $\angle B$ cannot equal $\angle A$ and cannot be less than it. Q.E.D.

120. Exterior Angles. The angle formed by producing a side of a triangle is called an **exterior angle** of the triangle.

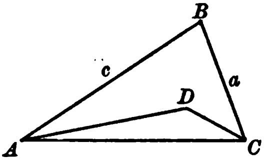
Thus, $\angle x$ and $\angle y$ are exterior angles of $\triangle ABC$.



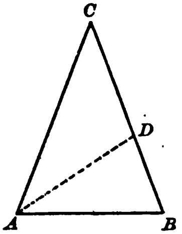
EXERCISES

1. If the exterior angles at the base of a triangle are equal, the triangle is isosceles.

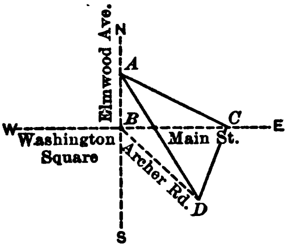
2. Given $\angle C > \angle A$, AD and DC bisectors of angles A and C. Prove that $AD > DC$.



3. Given that $\triangle ABC$ is isosceles, that $\angle BAC$ is twice $\angle C$ and that AD bisects $\angle BAC$; show that $\triangle DAC$ is isosceles.



4. In the figure, A is a telephone exchange and situated 800 ft. north of B. A telephone is located at C, $\frac{1}{2}$ mi. east of B, and another at D, $\frac{1}{2}$ mi. from B. Which telephone, C or D, is the farther, in a straight line, from the telephone exchange?



PROPOSITION XV. THEOREM

121. *If two triangles have two sides of the one respectively equal to two sides of the other, but the included angle of the first greater than the included angle of the second, then the third side of the first is longer than the third side of the second.*

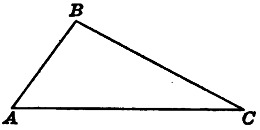


FIG. 1

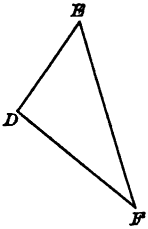


FIG. 2

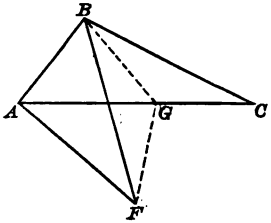


FIG. 3

Given $AB = DE$, $BC = EF$ and $\angle B > \angle E$.

To prove that $AC > DF$.

Proof. 1. Apply $\triangle DEF$ to $\triangle ABC$ so that DE coincides with AB, as in Fig. 3. Sec. 80, VII

2. Let BG bisect $\angle CBF$. Sec. 98

3. Then $\triangle FBG$ is congruent to $\triangle GBC$. Sec. 92

4. And $GF = GC$. Sec. 94

5. $AG + GF = AG + GC = AC$.
Adding AG to both members of (4). Sec. 78, II

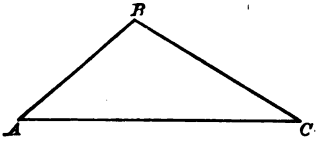
6. But, $AG + GF > AF$. Why?

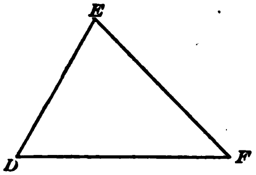
7. $\therefore$ $AC > AF$, or $AC > DF$. From step (4). Q.E.D.

NOTE.—The figure above was drawn with the point F below AC. It might also fall on AC or above AC, according to the length of EF. Draw the figures and give the proofs for these two cases.

PROPOSITION XVI. THEOREM

122. *If two triangles have two sides of the one respectively equal to two sides of the other, but the third side of the first greater than the third side of the second, then the included angle of the first is greater than the included angle of the second.*





Given $\triangle ABC$ and $\triangle DEF$ with $AB = DE$, $BC = EF$ and $AC > DF$.

To prove that $\angle B > \angle E$.

Proof. 1. $\angle B$ is either greater than, equal to or less than $\angle E$. No other possibility

2. If $\angle B = \angle E$, then $\triangle ABC$ is congruent to $\triangle DEF$, and $AC = DF$; but this is contrary to hypothesis. Sec. 92

3. If $\angle B < \angle E$, then $AC < DF$; but this is contrary to hypothesis. Sec. 121

4. $\therefore \angle B > \angle E$.

Because it is not equal to, nor less than, $\angle E$. Step (1). Q.E.D.

EXERCISES

1. If two unequal straight lines bisect each other obliquely, how many pairs of equal triangles are formed by connecting their end points?

2. How many equal triangles are there if the straight lines bisect each other perpendicularly?

3. If the equal sides of an isosceles triangle be prolonged beyond their common point, so that the prolongations are equal, the lines joining the ends of the prolongations to the ends of the base are also equal.

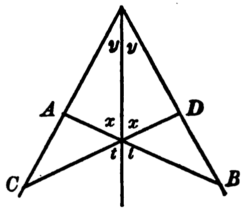
4. The extension of the bisector of one of two vertical angles bisects the other.

5. When two straight lines intersect, the bisectors of the four angles form two straight lines at right angles.

6. If two equal angles have the same vertex, and if their bisectors form a straight line, the angles are vertical angles.

7. Three straight lines intersect in a point. Prove that the sum of any three non-adjacent angles is 180°.

8. Two line segments comprised between the sides of an angle are equal if they intersect on the bisector and make the same angles with it. (In the figure prove that $AB = CD$.)



9. The median drawn from any vertex of a triangle is smaller than half the sum of the sides meeting at that vertex.

Suggestion. Draw $DE = BD$.

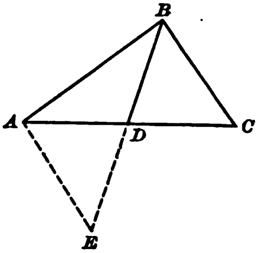
Then, $AB + AE > BE$,

or $AB + AE > 2\,BD$,

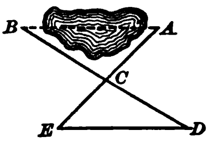
and $\frac{AB + AE}{2} > BD$. (1)

$\triangle ADE$ is congruent to $\triangle BDC$. Secs. 86, 92

$\therefore AE = BC$. Substitute BC in (1).

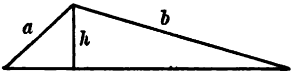


10. To find the length AB as indicated, a surveyor selected a convenient point C and measured AC and CB; he then laid off $CD = BC$, $CE = AC$ and measured ED. Show that $ED = AB$.



11. The altitude from any vertex of a triangle is less than half the sum of the two sides meeting at that vertex.

Suggestion. (1) $h < a$. Why?
(2) $h < b$. Why?
(3) Add (1) and (2) and divide by 2.

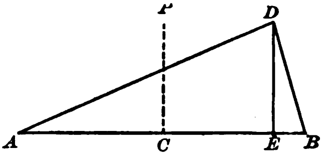


12. The proposition: Every point not on the perpendicular bisector of a line segment is unequally distant from its extremities, will now be proved in a new way.

Suggestion. In the figure, CP is the perpendicular bisector of AB, and $DE \perp AB$.

Therefore, $AE > EB$. Sec. 27

Which is the longer, DA or DB? Sec. 117, III



13. The sum of the three altitudes of a triangle is less than the sum of the three sides.

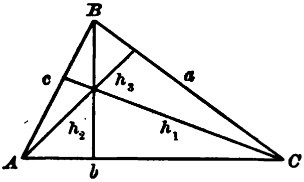
Suggestion. By Ex. (11),

(1) $h_1 < \frac{a+b}{2}$.

(2) $h_2 < \frac{a+c}{2}$.

(3) $h_3 < \frac{b+c}{2}$.

(4) Add (1), (2) and (3).



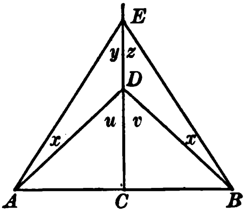
NOTE. — The symbol "h_1" is read: "h sub one," or simply "h one," and the others are read similarly.

14. In the figure, CE is the perpendicular bisector of AB, and $DA = 5$ in. What is the length of BD?

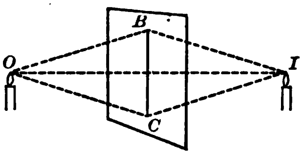
15. Given further that $EB = 7$ in. What is the length of EA?

16. Given further that the angles marked x are equal. Show that

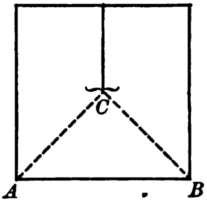
$\angle y = \angle z$; and that $\angle u = \angle v$.



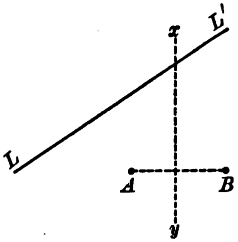
17. The image of an object reflected in a plane mirror is as far back of the mirror as the object is in front of it. In the figure, *OI* is a perpendicular bisector of *BC*. Compare *OB* with *BI*; *OC* with *CI*. Give the reason for your conclusions.



18. A chandelier hangs in the center of a room. If the walls are vertical and the floor level, compare *BC* with *AC*.



19. *A* and *B* are two towns and *LL'* is a railroad. Find a place for a station on *LL'* which will be equally distant from *A* and *B*. How must the line *xy* be drawn?



20. The perpendiculars erected to the sides of an angle, at points equidistant from its vertex, meet on the bisector of the angle.

HISTORICAL NOTE

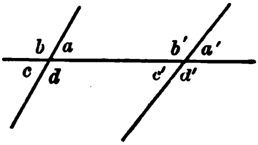
Euclid

The foregoing theorems and problems with their demonstrations sufficiently illustrate the peculiar kind of thinking and the logical sequence of facts found in geometry. Although the knowledge of some of the properties of figures that we have studied is as old as civilization, the Greek scholars were the first to establish most of them. For example, Thales of Miletus, about 600 B.C., is said to have proved that vertical angles are equal, that the base angles of an isosceles triangle are equal, that a diameter bisects a circle, and that certain triangles are congruent. Either Pythagoras of Samos (about 580 B C.), or his pupils, discovered many more of these properties, but all were scattered through the records and traditions of the numerous schools of philosophy and gave little evidence of their value.

About 300 B.C., there lived in Alexandria a Greek, named Euclid, who wrote the first work presenting elementary geometry essentially as we know the subject to-day. *The Elements*, as the work was called, contained thirteen rolls, or books, seven of which treated of incommensurables, theory of numbers, and other topics not now commonly included in elementary geometry. This remarkable specimen of logic, built upon twelve formulated axioms and known as Euclidean Geometry, has been subject to the scrutiny of critics of twenty-three centuries; other arrangements of propositions have been considered better by some, other sets of axioms have served as a basis for various "non-Euclidean" geometries in modern higher mathematics, but no other system equally simple and fundamental has yet been devised for teaching the elementary properties of plane figures.

Of Euclid himself little is known, but Proclus, a mathematician of the fifth century A.D., tells us that Euclid was born at Athens and came to Egypt in 300 B.C., in the reign of the first Ptolemy, to teach in the University of Alexandria. It was here that he wrote *The Elements*, on which rests his great fame. King Ptolemy was impressed by the genius of Euclid, and is said to have asked him on one occasion if geometry could not be learned in an easier way than by the mastery of his *Elements*, which query provoked the well-worn proverb: "There is no royal road to Geometry." His character is quaintly described by Pappus, who states that he was distinguished by fairness and kindness of disposition, especially toward any one who could do aught to further mathematical science. That he was also impatient with those who could not see value in the discovery of truth for truth's sake may be inferred from the incident of the lazy pupil. For, it is related that when a complaining pupil asked "What do I get by all this work?" Euclid called his slave and said "Give this boy some coppers, since he must make profit out of what he learns."

123. Corresponding Angles. When two straight lines are cut by a third line, the pairs of angles which lie on the same side of the cutting line, and on the same side of the given lines, are called **corresponding angles.**



Thus a and a' are a pair of corresponding angles; likewise, b and b', c and c', d and d'.

124. The cutting line is called a **transversal** or a **secant.**

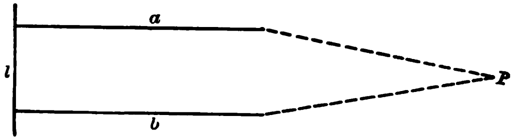
EXERCISES

1. Draw two lines and a transversal. Letter the angles and point out pairs of corresponding angles.

2. Draw two intersecting lines, l and m. At some point of m, draw a line q so that the angle between q and m appears to equal the corresponding angle between l and m.

PROPOSITION XVII. THEOREM

125. *Two straight lines perpendicular to the same line cannot meet.*

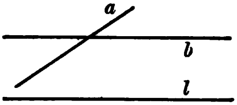


For, if a and b, each perpendicular to l, could meet in P, there would be two perpendiculars from P to l, contrary to Sec. 108.

126. Parallel Lines. Two lines in the same plane that cannot meet, however far extended, are called **parallel lines.**

Consequently, *Perpendiculars to the same straight line are parallel.*

127. Postulate. *Through a given point, only one line can be drawn parallel to a given line.*

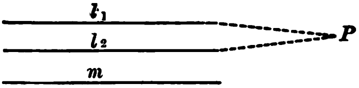


That is, a and b cannot both be parallel to l.

Consequently, *If a straight line cuts one of two parallel lines, it must cut the other if sufficiently extended.*

PROPOSITION XVIII. THEOREM

128. *If two lines are parallel to a third line, they are parallel to each other.*



Given l_1 and l_2, each parallel to m.

To prove that $l_1 \parallel l_2$.

Proof. If l_1 meets l_2 at any point, as P, then there are two lines through P parallel to m, contrary to Sec. 127.

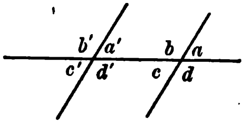
Therefore, $l_1 \parallel l_2$.

EXERCISES

1. Extend all of the sides of a square; select the pairs of parallel lines. Why are they parallel?
2. Select two pairs of corresponding angles.

PROPOSITION XIX. THEOREM

129. *If two straight lines are situated so that when cut by a transversal the angles of one corresponding pair are equal, the angles of every corresponding pair are equal.*



Given $\angle a = \angle a'$.

To prove that $\angle b = \angle b'$, $\angle c = \angle c'$ and $\angle d = \angle d'$.

Proof. 1. We know that $\angle a + \angle b = 180°$.

$\therefore \angle b = 180° - \angle a$. Sec. 78, III

2. Similarly, $\angle a' + \angle b' = 180°$.

$\therefore \angle b' = 180° - \angle a' = 180° - \angle a$. $\angle a = \angle a'$. Given

3. From (1) and (2), $\angle b = \angle b'$. Sec. 78, I

4. In the same way, prove that $\angle c = \angle c'$, also $\angle d = \angle d'$.

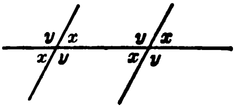
Q.E.D.

130. COROLLARY. $\angle a = \angle c'$, and $\angle a' = \angle c$.
$\angle d = \angle b'$, and $\angle d' = \angle b$.

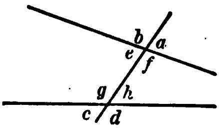
For, 1. $\angle a = \angle c$. Why?
2. $\angle c = \angle c'$. Sec. 129
3. $\therefore \angle a = \angle c'$. Sec. 78, I

Similarly, prove each of the others.

131. **Summary.** If a transversal cuts two lines so as to make one pair of corresponding angles equal, the eight angles are made up of two sets of four equal angles each, as shown in the figure.



132. When two straight lines are cut by a third, the angles formed are named as follows:



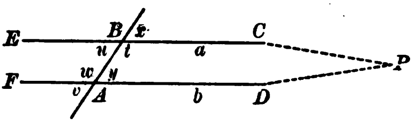
Using the notations of the figure, angles a, b. c, d are called **exterior angles**; angles a and c, also angles b and d, are called **alternate exterior angles.** Angles e, f, g, h are called **interior angles.** Angles f and g, also e and h, are called **alternate interior angles.**

PROPOSITION XX. THEOREM

133. *If two straight lines are situated so that when cut by a transversal,*

(I) *the angles of a corresponding pair are equal, the lines are parallel;*

(II) *the angles of a corresponding pair are unequal, the lines are not parallel.*



To prove I.

Given lines a and b with transversal AB and $\angle x = \angle y$.

Proof. 1. Suppose that a and b meet at P.

2. $\angle u = \angle v$; $\angle t = \angle w$. Since $\angle x = \angle y$. Secs. 129, 130

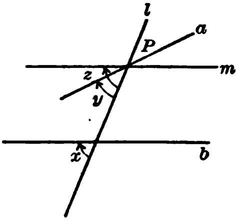
3. Suppose the strip of the plane $DABC$ turned through an angle of 180°, and placed upon the strip $FABE$ so that the ends AB of the two strips coincide, but the $\angle t$ has its vertex brought to A. Since $\angle t = \angle w$, BC will coincide with AF. Similarly, AD will coincide with BE.

4. Then, if BC and AD produced meet at P on the right, EB and AF produced will meet at some point Q on the left.

5. But it is not possible that the straight lines EC and FD should intersect in two distinct points P and Q. Sec. 17. Hence, they cannot intersect at all; that is, they are parallel.

To prove II.

Given lines a and b, making with l corresponding angles that are unequal. Let x denote the greater and y the smaller of these angles.



To prove that a is not parallel to b.

Proof. 1. Assume a line m through P, making $\angle z = \angle x$.
2. Then, $m \parallel b$. Sec. 133
3. $\therefore$ a cannot be parallel to b, and will meet it. Sec. 127

Q.E.D.

134. Cor. 1. *If a transversal is perpendicular to one of two parallels, it is perpendicular to the other also.*

For, if not, the corresponding angles would be unequal and the lines would not be parallel.

135. Cor. 2. *When two parallel lines are cut by a transversal, the two angles of each pair of alternate interior angles are equal, and the two angles of each pair of alternate exterior angles are equal. Also the angles of each pair of corresponding angles are equal.*

136. Cor. 3. *When two parallel lines are cut by a third line, the sum of the angles in each pair of interior angles on the same side of the transversal is a straight angle; and the sum of the angles in each pair of exterior angles on the same side of the transversal is a straight angle.*

Show this by use of the figure of Sec. 129.

137. Parallels may be drawn by means of a square as illustrated in Fig. 1; AB and CD are parallel, since they make the same angles with AC. They may be drawn similarly by means of a bevel, as illustrated in Fig. 2.

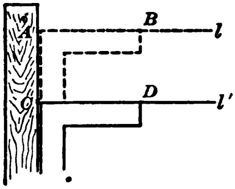


FIG. 1

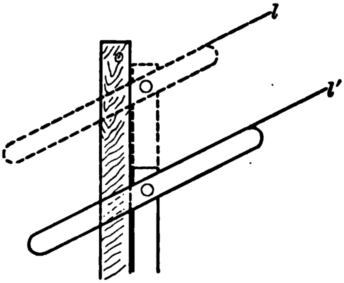


FIG. 2

The illustrations above show how to draw lines that are approximately parallel by means of the instruments named. We shall see later how to do the same thing by means of the straightedge and compasses, which are the only instruments customarily supposed available in elementary geometry.

In all that follows, we may suppose any lines drawn or points marked that we know exist (such as perpendiculars, parallels, bisectors of segments and angles, mid-points and the like), even though we do not as yet know any method of doing so approximately by means of instruments. In actual figures, the marks representing such points and lines may be inserted carefully freehand or by means of such instruments as are available, aided by the judgment of the eye. For example, a perpendicular from a point to a line may be drawn either freehand or by means of a ruler, the eye judging in each case when the angle formed is about a right angle.

EXERCISES

1. Prove by the proposition of Sec. 125 that no triangle can have two right angles.

2. A chandelier is perpendicular to the ceiling of a room. Prove that the chandelier is parallel to the lines at the corners where the walls of the room come together.

PROPOSITION XXI. THEOREM

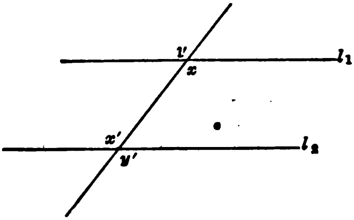
138. *Two straight lines are parallel, if any one of the following relations exist between the angles formed by a transversal:*

(I) *Two alternate interior angles are equal.*

(II) *Two alternate exterior angles are equal.*

(III) *The sum of two interior angles on the same side of the transversal is* 180°.

(IV) *The sum of two exterior angles on the same side of the transversal is* 180°.

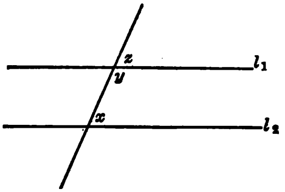


Proof of I.

1. $\angle x = \angle x'$. Given
2. But, $\angle x = \angle y$. Why?
3. $\therefore \angle x' = \angle y$. Why?
4. Therefore $l_1 \parallel l_2$. Sec. 133. Q.E.D.

Proof of II. Follow the method of I, in the figure above.

Proof of III.



1. $\angle x = 180° - \angle y$. Given
2. $\angle z = 180° - \angle y$. Why?

3. $\therefore \angle x = \angle z$. Why?

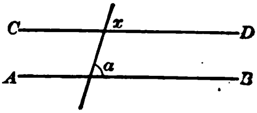
4. $\therefore l_1 \parallel l_2$. Sec. 133, I

Q.E.D.

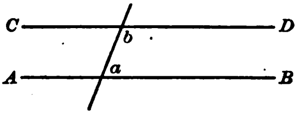
Proof of IV. Follow the method of III.

EXERCISES

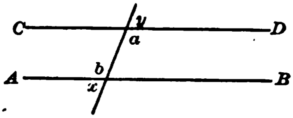
1. If angle $a = 37°$, and AB is parallel to CD, how many degrees in angle x?



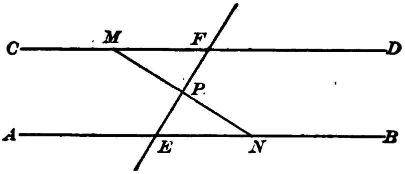
2. If angle $a = 41°$, and angle $b = 138°$, are lines AB and CD parallel? Why?



3. If AB and CD are parallel, show that the bisectors of angles a and b are parallel. Show that the bisectors of angles x and y are parallel.



4. If AB and CD are parallel, and if P is the midpoint of EF, show that any line through P and terminated by the parallels is bisected by P.



5. A line parallel to the base of an equilateral triangle, and cutting its sides, cuts off an equilateral triangle.

6. Every line cutting the sides of an isosceles triangle, and parallel to its base, cuts off an isosceles triangle.

7. Every line cutting the sides of an isosceles triangle, and parallel to its base, is bisected by the altitude of the triangle.

8. Every line cutting the hypotenuse of a right triangle and one side, and parallel to the base cuts off a right triangle.

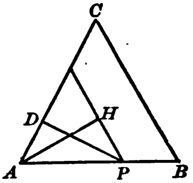
9. If from any point in the bisector of an angle, a line be drawn parallel to one side, and meeting the other, prove that the triangle so formed is isosceles.

10. If two sides of a triangle be produced beyond their common point, each by a distance of its own length, prove that the line joining the end points of these extensions is parallel to the third side of the triangle.

11. Perpendiculars drawn from the extremities of one side of a triangle to the median to that side, or to the median produced, are equal and parallel.

12. The bisector of an exterior angle at the vertex of an isosceles triangle is parallel to the base (and conversely).

13. If from any point of the base of an isosceles triangle ABC, a perpendicular to one side and a parallel to the other be drawn, and if a perpendicular AH be erected on the parallel, as indicated in the figure, show that $AH = PD$.



14. When a ray of light in passing from B to C is reflected from a smooth surface, like a mirror as at A, Fig. 1, $\angle x$, at which the ray meets the surface, is called the **angle of incidence**, and the $\angle y$, at which it leaves the surface, is called the **angle of reflection.** It is known that the two angles are equal. In the periscope of Fig. 2, what must be the value of $\angle$s x in order that the mirrors may reflect the ray of light along the path indicated?

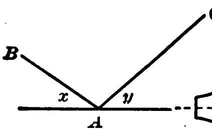


FIG. 1

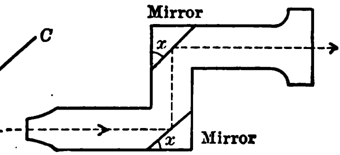


FIG. 2

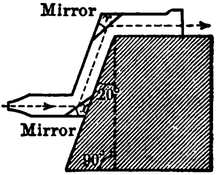
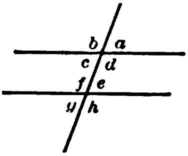


FIG. 3

15. In the periscope of Fig. 3 what must be the value of $\angle$s x in order that the mirrors may reflect the light as shown?

16. Using the letters to denote the angles marked, show that if any one of the following relations hold, they all hold.

(1) $a = e$.
(2) $a = g$.
(3) $b = h$.
(4) $c = g$.
(5) $c = e$.
(6) $d = f$.
(7) $d = h$.
(8) $b = f$.
(9) $a + h =$ a straight angle.
(10) $c + f =$ a straight angle.
(11) $d + g =$ a straight angle.
(12) $b + e =$ a straight angle.
(13) $b + g =$ a straight angle.
(14) $d + e =$ a straight angle.
(15) $a + f =$ a straight angle.
(16) $c + h =$ a straight angle.



PROPOSITION XXII. THEOREM

139. *Two angles whose sides are respectively parallel are either equal or supplementary.*

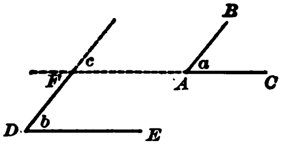


FIG. 1

Given acute angles a and b, having $AC \parallel DE$, and $AB \parallel DF$.

To prove that $\angle a = \angle b$.

Proof. 1. Produce AC and DF. They will intersect. Why?

2. Then, $\angle a = \angle c$ and $\angle c = \angle b$. Why?

3. Hence, $\angle a = \angle b$. Why?

Given one of the angles to be acute and the other obtuse.

To prove that they are supplementary.

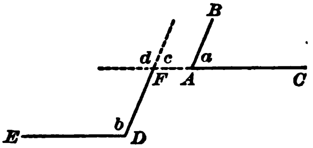
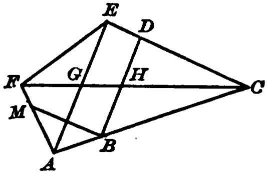


FIG. 2

State the proof, using figure 2.

Given both angles to be obtuse. Prove them equal.

140. It is often convenient in designating the corresponding sides of angles to use the terms *right* and *left*.



The side of an angle which lies to the right when the angle is viewed from the vertex is called the *right side* of the angle, and the side lying to the left, the *left side* of the angle.

In the figure, name the right side of $\angle AED$; of $\angle EFG$. Similarly, name the left side of $\angle GHD$; of $\angle MFG$; of $\angle GEF$; of $\angle DHC$.

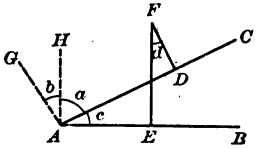
141. With this convention, the ambiguity in Proposition XXII can be removed by re-stating it thus:

Two angles are equal if their right sides are parallel and their left sides are parallel.

Two angles are supplementary if the right side of each is parallel to the left side of the other.

PROPOSITION XXIII. THEOREM

142. *If the right sides of two angles are perpendicular and their left sides are perpendicular, the angles are equal.*



Given $\angle BAC$ and $\angle EFD$, having $EF \perp AB$ and $DF \perp AC$.

To prove that $\angle BAC = \angle EFD$.

Proof. 1. Through A, assume $AH \parallel EF$, and $AG \parallel DF$.

2. Then $\angle b = \angle d$. Sec. 141

3. But $HA \perp AB$, and $GA \perp AC$. Sec. 134

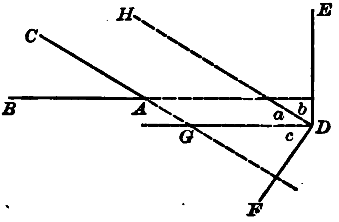
4. Then $\angle a + \angle b = \underline{\lfloor R}$, and $\angle a = \angle c = \underline{\lfloor R}$. Why?

5. Hence, $\angle a + \angle b = \angle a + \angle c$. Sec. 78, I

6. $\therefore \angle b = \angle c$. Sec. 78, III

7. $\therefore \angle c = \angle d$. From (6) and (2), Sec. 78, I. Q.E.D.

143. Corollary. *If two angles are so situated that the right side of each is perpendicular to the left side of the other, the angles are supplementary.*



Given $ED \perp AB$ produced, and $FD \perp AC$ produced.

To prove that $\angle FDE + \angle BAC = 2\underline{|R}$'s.

Proof. 1. Assume $DH \parallel AC$, and $DG \parallel AB$. Sec. 98

2. Then $\angle a = \angle BAC$. Sec. 141

3. But $\angle b + \angle a + \angle c = \angle EDF$.

4. $\therefore \angle b + \angle a + \angle c + \angle a = \angle BAC + \angle FDE$.
Adding (2) and (3)

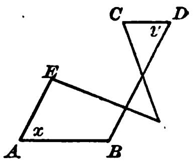
5. But $\angle b + \angle a = \underline{|R}$.

6. And $\angle c + \angle a = \underline{|R}$. Why?

7. $\therefore \angle b + \angle a + \angle c + \angle a = 2\underline{|R}$'s. Adding (5) and (6)

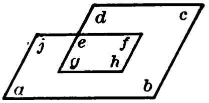
8. $\therefore \angle BAC + \angle FDE = 2\underline{|R}$'s. From (7) and (4). Q.E.D.

EXERCISES



1. Given AB parallel to CD, AE parallel to BD, compare angles x and y.

2. If the lines in the second figure that seem to be parallel represent parallel lines, name all the angles equal to a; all those equal to b.



3. If a triangle has its sides respectively parallel to the sides of an isosceles triangle, the first triangle is also isosceles.

144. Preparatory.

1. Cut out a paper triangle like ABC in Fig. 1.
2. Tear off the angles a, b, c as in Fig. 1.
3. Place them with vertices together as in Fig. 2.

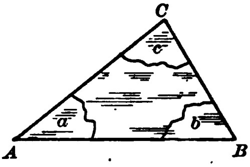


FIG. 1.

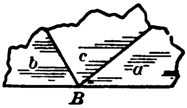
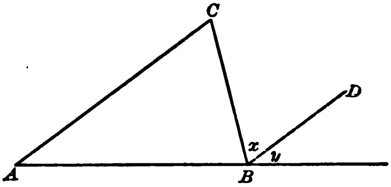


FIG. 2.

4. What kind of angle does their sum seem to form?
5. Try this with a triangle of a different shape.

PROPOSITION XXIV. THEOREM

145. *The sum of the angles of any triangle equals two right angles.*



Given $\triangle ABC$.

To prove that $\angle A + \angle B + \angle C = 2\,\underline{|R}$'s.

Proof. 1. Assume $BD \parallel AC$. Sec. 98

2. Then, $\angle x = \angle C$. Why?

3. Also $\angle y = \angle A$. Why?

4. But $\angle B + \angle x + \angle y = 2\,\underline{|R}$'s.

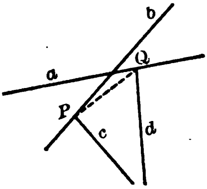
5. $\therefore \angle B + \angle C + \angle A = 2\,\underline{|R}$'s.

From steps (2), (3), (4), Sec. 78, I. Q.E.D.

146. Cor. 1. *If any side of a triangle be produced, the exterior angle formed is the sum of the two opposite interior angles, and greater than either of them.*

147. Cor. 2. *A triangle cannot have more than one right angle, or more than one obtuse angle.*

148. Cor. 3. *In a right triangle, the sum of the two acute angles is a right angle.*



149. Cor. 4. *Two lines perpendicular respectively to two intersecting lines must meet if extended sufficiently.*

Suggestion. $d \perp a$ and $c \perp b$.

Connect P and Q, and apply Sec. 138, III.

EXERCISES

1. In triangle ABC, angle A equals 70°, angle B equals 60°; what is the third angle? Which is the longest side of the triangle? The shortest?

2. In triangle ABC, angle C equals 40°, the exterior angle at B equals 100°. Find angles A and B of the triangle. Which is the longest side? The shortest?

3. How many degrees in each angle of an equilateral triangle?

4. The angle at the apex of an isosceles triangle is 80°. How many degrees in each base angle?

5. How many degrees in each acute angle of an isosceles right triangle?

6. Prove that the altitudes from two vertices of a triangle form one angle equal to the angle of the triangle at the third vertex, and another supplementary to it.

7. The altitude from an extremity of the base of an isosceles triangle makes an angle with the base equal to half the angle at the apex.

Suggestion. Assume the altitude from the apex.

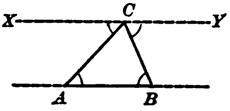
8. Construct an angle of 60°.

Suggestion. Construct an equilateral triangle. Page 22, Ex. 3.

9. How many degrees in the angles between the sides of a square and the line joining the opposite corners? Prove it by Ex. 5.

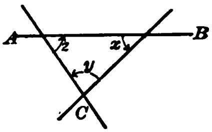
10. Given a square, construct an angle of 45°.

11. In the figure $XY \parallel AB$. Use this line to prove Prop. XXIV, Sec. 145.

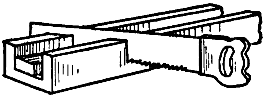


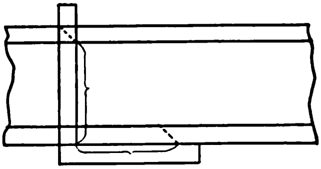
12. A line may be reversed by turning it through a straight angle. Show that side AB of triangle ABC is reversed by turning it through the three angles x, y, z in order, in the direction indicated by the arrows.

What proposition does this prove?



13. In a "miter box" used to saw picture molding, a slit in the box holds the saw at an angle of 45° with the molding. Show how the place for the slit can be marked with the aid of a square.



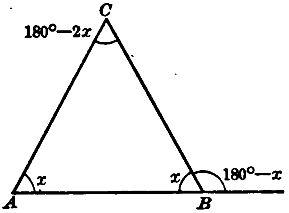


14. Prove that two equilateral triangles are equal if an altitude of one equals an altitude of the other.

15. Two isosceles triangles are equal if the base and opposite angle of one are equal respectively to the base and opposite angle of the other.

16. The bisectors of the interior angles on the same side of a transversal of two parallels are perpendicular to each other.

17. Show that in an isosceles triangle, an exterior angle at the base exceeds a right angle by half the angle at the apex.



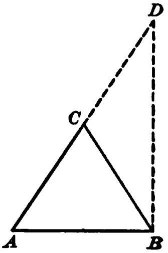
Suggestion. According to the figure, the proof is made, if we show that

$$\angle 180^\circ - x = 90^\circ + \tfrac{1}{2}(180^\circ - 2x).$$

This is readily done.

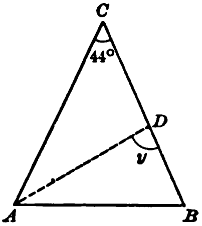
18. Prove Ex. 17 by assuming a perpendicular CP from C to AB and a parallel to CP through B.

19. In the figure, $\triangle ABC$ is isosceles and $CD = AC$. Prove that $DB \perp AB$.



Suggestion. What kind of triangle is $\triangle BCD$? Compare $\angle ABD$ with $\angle A + \angle D$. Each is what part of a straight angle?

20. In the figure, $\triangle ABC$ is isosceles and $\angle C = 44°$; AD is the bisector of $\angle A$; find $\angle y$.

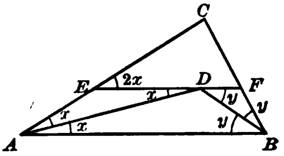


Suggestion. Find the sum of angles A and B. Then, the sum of $\frac{1}{2}\angle A$ and $\angle B$.

21. If the perpendiculars drawn from a point in the base of an isosceles triangle to its sides are equal, the point is the midpoint of the base.

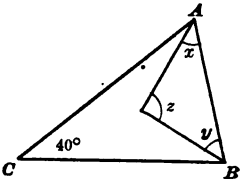
22. Any exterior angle of a triangle is greater than either interior angle not adjacent to it.

23. ABC is any triangle; AD and DB bisect angles A and B. $EF \parallel AB$. Prove that $EF = AE + BF$.



24. An angle of a triangle is 40°. How large is the angle between the bisectors of the other two angles?

Suggestion. $\angle A + \angle B = ?$
$\angle x + \angle y = ?$
$\therefore \angle z = ?$



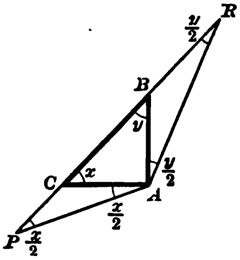
25. What is the result in Ex. 24, if the given angle is d degrees?

26. If the hypotenuse of a right triangle is produced beyond each vertex by a length equal to the side ending in that vertex, the lines joining the vertex of the right angle to the extremities of the produced parts inclose an angle of 135°.

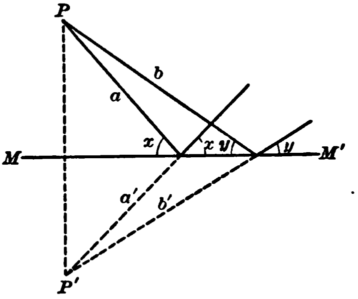
Suggestion. 1. In the figure, show why the angles that are lettered alike are equal.

2. In $\triangle BAC$, $\frac{\angle x + \angle y}{2} = ?$

3. $\therefore \angle PAR = \angle A + \frac{\angle x}{2} + \frac{\angle y}{2} = ?$



27. In the figure a and b are rays of light from P reflected by the mirror MM' so as to make angles equal as indicated in the figure. Show that the extensions of their reflected rays, a' and b', back of the mirror will have a common point P' (called the image of P) lying on the perpendicular from P to the mirror, and the same distance from the mirror as P.



Suggestion. Let a' cut the perpendicular at P'. $a' = a$. Why? Show by congruent triangles that b' goes through P'.

150. Quadrilateral. A portion of a plane bounded by four straight lines is called a **quadrilateral.**

1. **Sides.** The four lines are the **sides** of the quadrilateral.

2. **Vertices.** The four points in which the sides meet are the **vertices** of the quadrilateral.

3. **Adjacent and opposite sides and vertices.** If two sides meet in a common vertex they are called adjacent; otherwise, **opposite.**

If two vertices are the extremities of the same side they are called **adjacent**; otherwise, **opposite.**

4. **Angles.** The angle formed by two adjacent sides is called an angle of the quadrilateral.

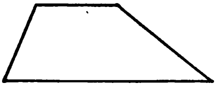
5. **Diagonals.** A line connecting two opposite vertices is called a **diagonal** of the quadrilateral.

6. **Perimeter.** The distance around a quadrilateral, or the sum of the lengths of its sides, is called its **perimeter.**

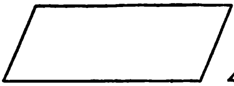
151. Special Varieties of Quadrilaterals.

1. A quadrilateral with two sides parallel is called a **trapezoid.** If nonparallel sides are equal, the trapezoid is **isosceles.**

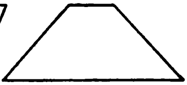
2. A quadrilateral with its opposite sides parallel is called a **parallelogram.**



Trapezoid



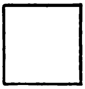
Parallelogram



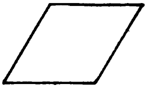
Isosceles Trapezoid

3. A parallelogram having four right angles is a **rectangle.** A rectangle having four sides equal is a **square.**

Rectangle



Square



Rhombus

4. A parallelogram having four oblique angles and its four sides equal is a **rhombus.**

152. Bases and Altitudes. The side on which a quadrilateral is supposed to stand is called its **base.**

In the case of parallelograms and trapezoids, it is customary to call one of the parallel sides the **lower base**, and the other the **upper base.**

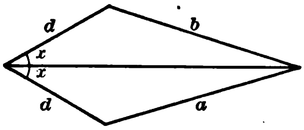
The **altitude** of a parallelogram or a trapezoid is the perpendicular distance between the lower and the upper base.

EXERCISES

1. A rectangle is a parallelogram. Why?

2. A piece of land in the shape of a quadrilateral has two adjacent sides equal, and the diagonal from their common point bisects the angle between them.

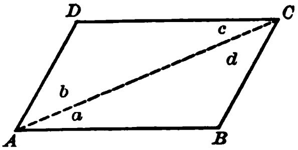
Prove that the diagonal divides the piece of land into two equal parts. Compare the sides a and b.



3. Prove that two parallelograms are congruent if two sides and the included angle of one are respectively equal to two sides and the included angle of the other.

PROPOSITION XXV. THEOREM

153. *The opposite sides of a parallelogram are equal, and the opposite angles are equal.*



Given parallelogram $ABCD$.

To prove $AB = DC$, and $AD = BC$; also $\angle A = \angle C$ and $\angle B = \angle D$.

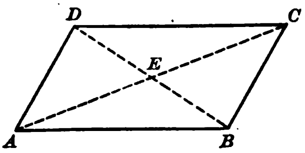
Proof. 1. Assume the diagonal AC.
2. $\angle b = \angle d$, and $\angle a = \angle c$. Why?
3. $\triangle ACD$ is congruent to $\triangle ACB$. Why?
4. $\therefore DC = AB$, $BC = AD$, $\angle B = \angle$ D. Why?
5. And $\angle C = \angle A$. Adding the equations in (2). Q.E.D.

154. Cor. 1. *Parallel lines are everywhere equidistant from each other.* (Assume perpendiculars between the parallels.)

155. Cor. 2. *The sum of the angles of any quadrilateral is four right angles.* (Assume a diagonal.)

Proposition XXVI. Theorem

156. *The diagonals of a parallelogram bisect each other.*



Given parallelogram $ABCD$ with diagonals AC and DB.

To prove that $DE = EB$, and $AE = EC$.

Proof. 1. In the triangles AEB and DEC, $\angle ACD = \angle CAB$, $\angle BDC = \angle DBA$. Sec. 135
2. $AB = CD$. Sec. 153
3. $\therefore DE = EB$, and $AE = EC$. Sec. 94

157. Corollary. *The diagonals of a square bisect each other at right angles.*

EXERCISE

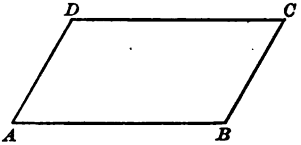
1. If two angles of a quadrilateral are supplementary, the other two are supplementary.

PROPOSITION XXVII. THEOREM

158. *If the opposite angles in a quadrilateral are equal, or if the opposite sides are equal, the quadrilateral is a parallelogram.*

I. **Given.** Quadrilateral $ABCD$, with $\angle A = \angle C$, and $\angle B = \angle D$.

To prove that $ABCD$ is a parallelogram.



Proof. 1. $\angle A + \angle B + \angle C + \angle D = 4 \underline{|R}$'s. Sec. 155

2. $2\angle A + 2\angle B = 4 \underline{|R}$'s. Given

3. $\therefore \angle A + \angle B = 2 \underline{|R}$'s. Why?

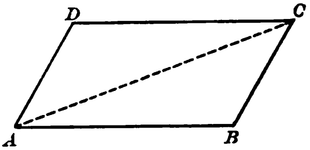
4. Then, $AD \parallel BC$. Sec. 138, III

5. Similarly, $AB \parallel DC$.

6. $\therefore$ $ABCD$ is a parallelogram. Sec. 151, 2

II. **Given** $AD = BC$, and $AB = CD$.

To prove that $ABCD$ is a parallelogram.



Proof. 1. Assume diagonal AC.

2. Then, $\triangle$ ABC and ADC are congruent. Why?

3. Hence, $\angle ACB = \angle DAC$.

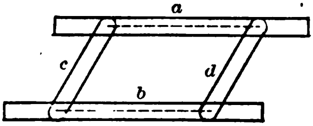
4. Likewise, $\angle DCA = \angle BAC$.

5. $\therefore BC \parallel AD$ and $AB \parallel DC$. Sec. 133, I

6. $\therefore$ $ABCD$ is a parallelogram. Def. ▱. Q.E.D.

EXERCISES

1. Parallels may be drawn by means of "parallel rulers" like the cut. The distances a and b are equal and the bars c and d are equal. How does the theorem show that the lines will be parallel in all positions of the rulers?

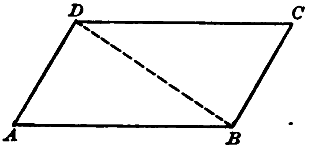


2. If from a point within an acute angle perpendiculars are drawn to the sides of the angle, the angle between the perpendiculars is the supplement of the given angle.

3. Discuss Exercise 2 in the case of an obtuse angle, first when the point is so situated that both perpendiculars meet the sides of the angle; second, when one of the perpendiculars meets a side produced.

PROPOSITION XXVIII. THEOREM

159. *If two opposite sides of a quadrilateral are equal and parallel, the figure is a parallelogram.*



Given. Quadrilateral $ABCD$, with $AB = DC$, and $AB \parallel DC$.

To prove that $ABCD$ is a parallelogram.

Proof. 1. Assume the diagonal DB. Sec. 98

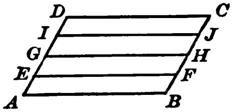
2. Then, $\triangle ABD$ is congruent to $\triangle BCD$. Why?

3 $\therefore \angle ADB = \angle DBC$, Why?

4. and $AD \parallel BC$. Sec. 133, I

$\therefore$ $ABCD$ is a parallelogram. Def. ▱. Q.E.D.

160. Cor. 1. *If in any parallelogram two opposite sides are divided into the same number (n) of equal parts, and the opposite points connected in order, by straight lines, n equal parallelograms are formed.*

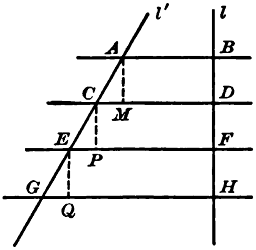


161. Cor. 2. *If a rectangle is treated as in Corollary 1, n equal rectangles are formed.*

162. Cor. 3. *If both pairs of opposite sides of a rectangle are treated as in Corollary 1, the length of the single divisions being equal on both pairs of sides, the smallest figures are all equal squares.*

Proposition XXIX. Theorem

163. *If three or more parallels intercept equal segments on one transversal, they intercept equal segments on every transversal.*



Given the ‖s AB, CD, EF and GH, making equal segments BD, DF and FH on transversal l.

To prove that $AC = CE = EG$ on transversal l'.

Proof. 1. Assume AM, CP, EQ, each parallel to l.

Then, $AM \parallel CP \parallel EQ$. Why?

2. Hence, $\angle MAC = \angle PCE = \angle QEG$. Sec. 135

3. $\triangle s\ AMC, CPE, EQG = \triangle s\ BDC, DFE, FHG$, respectively. Sec. 135

4. But $\angle BDC = \angle DFE = \angle FHG$. Why?
$\therefore \angle AMC = \angle CPE = \angle EQG$. Sec. 78, I

5. In ▱s $ABDM$, $CDFP$, $EFHQ$, $AM = BD$, $CP = DF$ and $EQ = FH$. Sec. 153

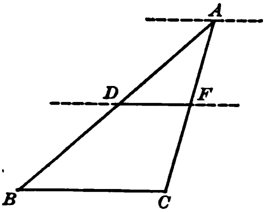
6. But, $BD = DF = FH$. $\therefore AM = CP = EQ$. Given

7. $\therefore$ $\triangle$s AMC, CPE, EQG are congruent. Sec. 97

8. $\therefore AC = CE = EG$. Q.E.D.

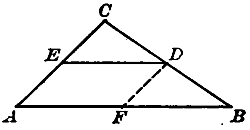
164. Cor. 1. *If a line parallel to one side of a triangle bisects one of the two remaining sides, it bisects the other side also.*

Suggestion. Through the vertex A, assume a line parallel to the base BC. Apply the theorem to the two transversals AB and AC.



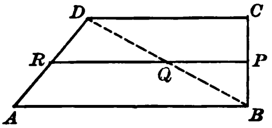
165. Cor. 2. *The line joining the mid-points of two sides of a triangle is parallel to the third side, and equal to half of it.*

Suggestion. Let D and E be the mid-points of the sides. Draw $DF \parallel CA$. Show that $\triangle$s CDE and BDF are congruent. Then $DF = CE = EA$. $AFDE$ is a parallelogram (Sec. 159). Then $ED = AF = FB$, and finally $AB = 2\,DE$.

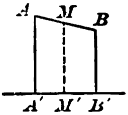


166. Cor. 3. (1) *The segment that joins the mid-points of the non-parallel sides of a trapezoid is parallel to the bases, and is equal to half the sum of the bases.* (2) *A line through the mid-point of one side and parallel to the bases passes through the mid-point of the other side.*

Suggestion. Let P, Q, R be the mid-points of CB, DB, AD respectively. By Cor. 2 and Sec. 128, QR and QP are both $\parallel AB$. $\therefore PQR$ is a straight line $\parallel AB$ By Cor. 2, $QR + QP = \frac{1}{2}AB + \frac{1}{2}DC$.



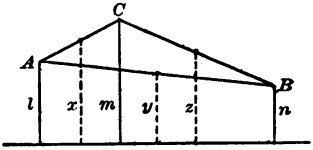
EXERCISES

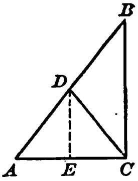


1. The sum of the distances of the end points of a segment, from any straight line not intersecting it, is twice the distance of the mid-point of the segment from that line.

2. The sum of the distances of the three vertices of a triangle from any straight line without the triangle is equal to the sum of the distances of the mid-points of the sides from that line.

Suggestion. $l + m = 2x$. Ex. 1
$l + n = 2y$. Ex. 1
$m + n = 2z$. Ex. 1
$\therefore 2l + 2m + 2n = 2(x + y + z)$. Why?
$\therefore l + m + n = x + y + z$. Why?





3. The median to the hypotenuse of a right triangle is half the hypotenuse.

Suggestion. Assume $DE \parallel BC$. Why is DE the perpendicular bisector of AC?

Compare DC with AD and with DB.

167. Polygon. A portion of a plane bounded by a closed broken line is called a **polygon.**

The terms side, vertex, adjacent sides or vertices, angles, diagonal, perimeter are applied to all polygons in the same sense as to the special varieties of polygons in Secs. 150, 151.

168. Kinds of Polygons. In some cases a special name is used to designate polygons of a certain number of sides. The most important names thus used are:

NUMBER OF SIDES	NAME	NUMBER OF SIDES	NAME
3	Triangle	6	Hexagon
4	Quadrilateral	8	Octagon
5	Pentagon	10	Decagon

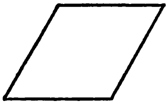
169. A polygon is **convex** if each of its angles is less than a straight angle.

A polygon is **concave** if it has an angle greater than a straight angle.

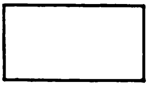
In what follows the polygons are always to be taken as convex unless otherwise stated.

170. A polygon is **equilateral** if all of its sides are equal.

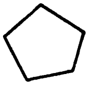
171. A polygon is **equiangular** if all of its angles are equal.



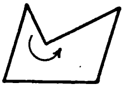
Equilateral



Equiangular



Convex



Concave

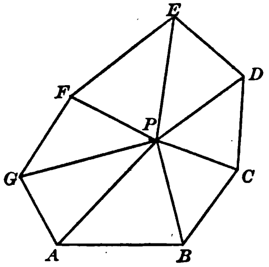
172. Regular Polygons. A polygon is **regular** if it is both equilateral and equiangular.

PROPOSITION XXX. THEOREM

173. *The sum of the interior angles of a polygon is equal to twice as many right angles as the polygon has sides, less four right angles.*

Given. A polygon having n sides.

To prove that the sum of the angles is $2n - 4$ right angles.



Proof. 1. Select any point P within the polygon $ABCD \cdots$ and connect it with each of the vertices, thus forming n triangles.

2. The sum of the angles of these triangles equals the sum of the angles of the polygon plus the sum of the angles at P.

3. The sum of the angles at P is $4\ \underline{|R}$'s.

4. The sum of the angles of the triangles is $2n\ \underline{|R}$'s.

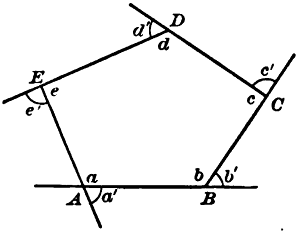
5. $\therefore$ the sum of the $\angle$s of the polygon is $(2n - 4)\ \underline{|R}$'s. Q.E.D.

174. COR. 1. *If the angles of a quadrilateral are equal, each is a right angle.*

175. COR. 2. *Each angle of a regular polygon of n sides is equal to* $\frac{2(n-2)}{n}$ *right angles.*

PROPOSITION XXXI. THEOREM

176. *The sum of the exterior angles of any polygon formed by extending each of its sides in succession, is four right angles.*



Given the polygon *ABCDE*, having *n* sides extended in succession.

To prove that the sum of the exterior angles is four right angles.

Proof. 1. Let $a, b \cdots e$ be the interior angles of the polygon, and $a', b' \cdots e'$ be the corresponding exterior angles. Then, $a + a' = 2$ right angles, $b + b' = 2$ right angles, and so on for each pair.

2. $\therefore$ the sum of all the interior and exterior angles in a polygon of n sides is $2n$ right angles; that is,

$$a + b + \cdots e + a' + b' + \cdots e' = 2n \text{ right angles.}$$

3. But, $a + b + \cdots e = 2n$ right angles $- 4$ right angles. Sec. 173

4. $\therefore$ $a' + b' + \cdots e' = 4$ right angles. Sec. 78, III. Q.E.D.

REVIEW EXERCISES

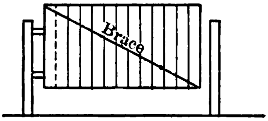
1. If one angle of a parallelogram is a right angle, all its angles are right angles.

2. If one angle of a parallelogram is 80°, how many degrees in each of the others?

3. If an angle of one parallelogram equals an angle of another, the other angles taken in order around the parallelograms are also respectively equal.

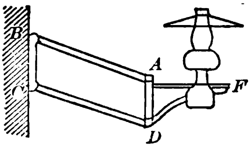
4. If two sides of a quadrilateral are parallel, and if a pair of opposite angles are equal, the figure is a parallelogram.

5. The line joining the middle points of two opposite sides of a parallelogram is parallel to the other two sides.



6. In the figure, the brace keeps the gate from sagging. On what proposition does this depend?

7. A lamp bracket is made as indicated in the cut. $CB = AD$, and $CD = AB$. There are movable joints at A, B, C, and D, and CB is fastened to a vertical wall. The lamp holder, AF, is at right angles to AD. Show that AF will remain horizontal as the bracket moves up or down.



8. If the diagonals of a parallelogram are equal, the figure is a rectangle.

9. The bisectors of the opposite angles of a parallelogram are parallel.

10. Two parallelograms are equal, if they have:

1. Two sides and the included angle respectively equal.
2. Two sides and the included diagonal respectively equal.
3. One side and the diagonals respectively equal.
4. One diagonal equal, and the angle between this diagonal and the adjacent sides respectively equal.

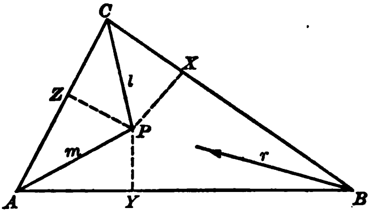
177. Concurrent Lines. Lines meeting in a point are said to be **concurrent.**

The following are simple cases of concurrent lines connected with the triangle.

1. *The bisectors of the angles of a triangle are concurrent; their common point is equidistant from the three sides of the triangle.*

Given $\triangle ABC$ with l, m, r the bisectors of $\angle s\ C$, A, B respectively.

To prove that l, m, r are concurrent.



Proof. 1. $\angle ACP + \angle PAC < 2$ $\underline{|R}$'s. Sec. 145

2. $\therefore$ l and m meet at some point as P. Sec. 138, III

3. P is equidistant from AC, AB and BC; that is, $PZ = PX = PY$. Sec. 111, I

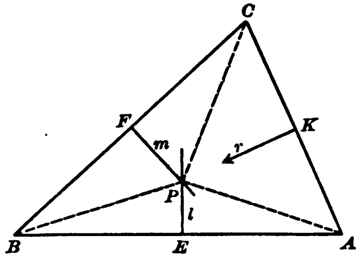
4. r, if sufficiently extended, must pass through P. Sec. 111, II

5. $\therefore$ l, m, r are concurrent, and their point of intersection is equidistant from AB, BC, AC. Step (3). Q.E.D.

2. *The perpendicular bisectors of the sides of a triangle are concurrent in a point equidistant from the vertices of the triangle.*

Given $\triangle ABC$, with l, m, r the perpendicular bisectors of AB, BC and AC respectively.

To prove that l, m, r are concurrent.



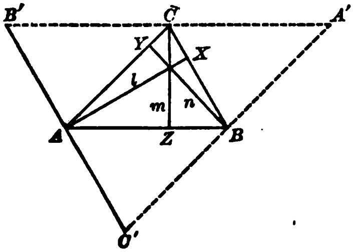
Proof. 1. l and m, if sufficiently extended, must meet, as at P. Sec. 149

2. P is equidistant from B, C and A. Why?

3. But r, if extended, must pass through P. Sec. 106

4. $\therefore$ l, m and r are concurrent, and their intersection is equidistant from A, B and C. Step (2). Q.E.D.

3. *The three altitudes of a triangle are concurrent.*



Given $\triangle ABC$, with l, m, n the altitudes from A, C, B respectively.

To prove that l, m, n are concurrent.

Proof. 1. Assume through A, B, C, lines parallel to BC, AC, AB respectively. They must meet and form a triangle, as $A'B'C'$. Why?

2. Then, $ABCB'$ and $AC'BC$ are ▱ with a common side. Why?

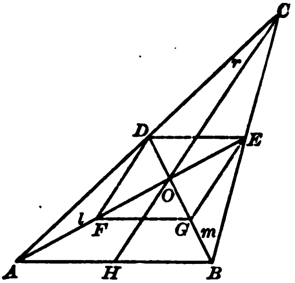
3. $\therefore B'A = AC'$. Sec. 153

4. Similarly, $B'C = CA'$, and $C'B = BA'$.

5. $\therefore l$, m, n are the perpendicular bisectors of $B'C'$, $B'A'$, $C'A'$ respectively. Sec. 134

6. $\therefore l$, m, r are concurrent. Sec. 177, 2. Q.E.D.

4. *The three medians of a triangle are concurrent; the common point is a trisection point of each median.*



Given $\triangle ABC$ and medians l, m, r to the sides BC, AC, AB respectively.

To prove that l, m, r are concurrent.

Proof. 1. l and m must meet. Why?

2. Let l and m intersect in O. Bisect AO and OB, as at F and G. Sec. 98

3. Draw FG, DE, DF, EG.

4. $DE \parallel AB$, and $FG \parallel AB$. Sec. 165

5. $\therefore DE \parallel FG$. Why?

6. Furthermore, DE and FG each $= \frac{1}{2} AB$. Sec. 165

7. Hence, $FGED$ is a parallelogram. Sec. 159

8. And O bisects FE and DG. Why?

9. Thus, O is a trisection point of l and m and is $\frac{2}{3}$ of the way from B and A.

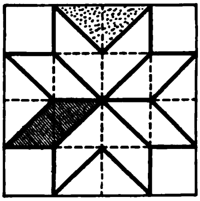
10. In the same way, r must cut m at a point $\frac{2}{3}$ of the way from B to D, or at the point O.

11. $\therefore l$, m, r are concurrent in a trisection point of each, two thirds of the way from each vertex of the triangle. Q.E.D.

178. The square, triangle, rectangle and parallelogram are the basis of many decorative designs.

EXAMPLES

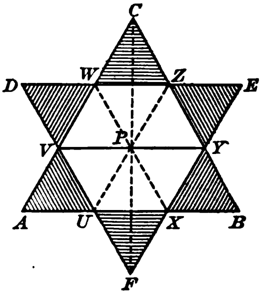
1. Prove that two equilateral triangles can be placed with one side in common, so as to form a parallelogram; three, so as to form a trapezoid; four, so as to form an equilateral triangle.



2. A square having each side divided into four equal parts is the basis of this design.

3. In the figure of Ex. 2 how many degrees in each angle of the shaded parallelogram?

4. In the above figure prove that the dotted portion is an isosceles triangle. How many degrees in each of its angles?



5. The basis of this design is the equilateral triangle ABC with each side trisected. Prove that WZ and VY are parallel to AB. Similarly, $XY \parallel AC$, and $UV \parallel BC$.

In the figure of Exercise 5:

6. Prove that the shaded triangles are all congruent.

7. Prove that $\triangle DEF$ is equilateral, and has its sides trisected by those of ABC.

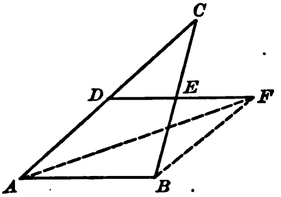
8. Prove that the perpendicular bisector of AB passes through C and F.

9. Connect A and E, D and B. Prove that if the figure is well drawn, these lines and CF will pass through one point P.

10. Describe a circle with P as a center, and PA as a radius. Prove that this circle should pass through A, F, B, E, C, D.

EXERCISES

1. If a rectangle and another parallelogram have equal bases and equal altitudes, the perimeter of the rectangle is shorter than that of the parallelogram.

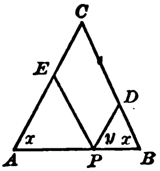


2. In the figure D and E are the mid-points of AC and BC respectively. BF is parallel to AC. Prove that $ABFD$ is a parallelogram.

3. The point of intersection of the diagonals of a rhombus is equidistant from the four sides of the rhombus.

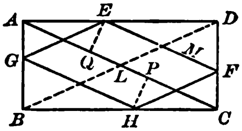
4. If from any point of the base of an isosceles triangle parallels be drawn to the other two sides, the parallelogram thus formed has a perimeter equal to the sum of the legs of the triangle. Prove for each of two different points of the base.

Suggestion. $\angle y = \angle x$.



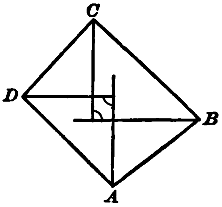
5. If in a rectangle, two lines parallel to one of the diagonals and equidistant from it are drawn, and the corresponding extremities of these parallels are joined, a parallelogram is formed, whose perimeter equals the sum of the diagonals of the rectangle.

Suggestion. 1. The distances HP and EQ are equal. Given
2. And $\triangle EAQ$ is congruent to $\triangle HPC$. Why ?
3. $\therefore AE = HC$.
4. Hence, $BH = ED$. Why ?
Similarly, $DF = GB$.
5. Also, $\angle GHB = \angle DEF$.
6. $\therefore \triangle HBG$ and $\triangle FDE$ are congruent. Why ?
7. $\therefore HFEG$ is a parallelogram. Why ?
8. $\triangle$ CLD, ADL, ALB, BLC are isosceles.
9. Therefore, Ex. 4 applies.



6. The diagonals of a rhombus bisect each other at right angles.

7. If the feet of the perpendiculars from the point of intersection of the diagonals to the sides of the rhombus be joined in order, a rectangle is formed.



8. The bisectors of the angles of a quadrilateral, produced until they intersect, form a quadrilateral whose opposite angles are supplementary.

179. Preparatory.

1. According to Fig. 1, what is the path of a point moving so as always to remain 6 in. from the given fixed point?

2. According to Fig. 2, what is the path of the center of the wheel, with radius 3 in., if it rolls along the straight line l?

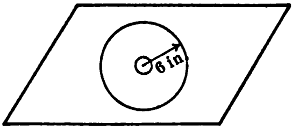


FIG. 1

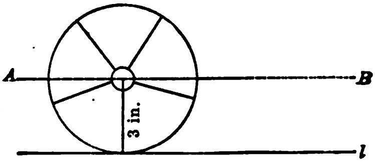


FIG. 2

3. State:

(1) Propositions, in two parts of Sec. 106.

(2) Propositions, Sec. 111, I, II, p. 47.

(3) The definition of the circle.

Of these three examples, each treats of a line or part of a line such that:

(*a*) *Every point of it satisfies a certain geometric condition,* and

(*b*) *Every other point does not satisfy the condition.*

180. Locus. The **locus** of all points satisfying a certain geometric condition is a figure such that every point of it satisfies the given condition, and that no other point satisfies the condition.

By use of the term **locus**, the propositions, Sec. 179, may be stated more tersely as follows:

1. Proposition (1) thus:

The locus of all points equidistant from the end points of a given segment is its perpendicular bisector.

2. Proposition (2) thus:

The locus of all points within an angle that are equidistant from the sides of the angle is the bisector of the angle.

3. The definition of a circle thus:

A circle is the locus of all points at a given distance from a fixed point.

The word "locus," from the Latin, means "place." It is pronounced "lōkus" (plural, lōçi, pronounced lōsī). The word "location" is from the same source.

EXERCISES

Draw the following loci without giving proofs:

1. The locus of all points 2 in. from a fixed point, P.

2. The locus of points 1 in. from a given straight line. (Note that the locus consists of two straight lines.)

3. The locus of points 1 in. from each of a pair of parallel lines 2 in. apart.

4. The locus of points $\frac{1}{2}$ in. outside of a circle of 1 in. radius.

5. The locus of points $\frac{1}{2}$ in. inside of a circle of $1\frac{1}{2}$ in. radius.

6. The locus of all points $1\frac{1}{2}$ in. from the intersection of the diagonals of a 2-inch square.

7. Draw a circle of radius 2 in. Mark a point, P, within it. Draw the locus of all points without the circle at distance $2\frac{1}{2}$ in. from P.

Show that:

8. The locus of all points at a given distance from a given straight line consists of two straight lines parallel to the given line and at the given distance from it.

9. The locus of all points equidistant from two parallel straight lines is a straight line parallel to the given straight lines and midway between them.

10. The locus of all points equidistant from two intersecting straight lines consists of the two straight lines formed by the bisectors of angles between the given straight lines.

181. Important Loci. The following loci are frequently used. Prove each statement.

1. *The locus of all points at a given distance from a given point is a circle with the given point as center and the given distance as the radius.*

2. *The locus of all points at a given distance from a given straight line is a pair of parallels, one on either side of the given line and each at the given distance from it.*

3. *The locus of all points that are equidistant from the sides of an angle is the bisector of the angle.*

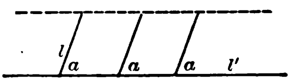
4. *The locus of all points that are equidistant from two given intersecting straight lines is the pair of bisectors of their included angles.*

5. *The locus of all points that are equidistant from two parallel straight lines is a parallel midway between them.*

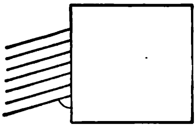
6. *The locus of the centers of all circles passing through two given points is the perpendicular bisector of the line joining the points.*

EXERCISES

1. If a straight line of given length moves with one extremity on a given line, and always makes the same angle with the given line and on the same side of it, what is the locus of its other extremity?



2. A straight line of given length moves around a square with one extremity always on the square, and so that all its positions are parallel. What is the locus of its other extremity? (It is understood, as always, that all of the figure is in the same plane.)



3. Solve the same problem if the line moves around any triangle; any parallelogram; any quadrilateral.

4. What is the locus of the mid-points of all segments with their extremities on two given parallels?

EXAMPLES OF THE USE OF LOCI

I. *To find the points of a given straight line that are at a given distance from a given point.*

Solution. 1. According to the problem, the desired points are among the points of the given straight line.

2. They are likewise among the points that are at the given distance from the given point.

3. But the locus made up of all such points is a circle with the given point as center and the given distance as radius.

4. Hence, the desired points are the points common to this circle and the given straight line.

Discussion. If the line and the circle should not intersect at all, there are no points satisfying the given conditions.

II. *To find a point equidistant from three given points A, B, C which are not in the same straight line (that is, prove Prop., Sec.* 177, 2 *by means of loci).*

Solution. 1. Since the point is to be equidistant from the three given points, it must be among the points that are equidistant from any two of them, as A and B.

2. But the locus made up of all such points is the perpendicular bisector of the segment AB (Sec. 180, 1). The desired point must be a point of this locus.

3. Similarly, the desired point is a point of the perpendicular bisector of AC.

4. Hence the desired point is the point of intersection of these two perpendiculars.

Discussion. If the three points had been given on the same straight line, the two bisectors would have been perpendicular to the same straight line, hence parallel, and therefore would not intersect. There is no point equidistant from three points lying on the same straight line.

The term "collinear" is often used for "lying on the same straight line."

The result proves also that through any three non-collinear points, one and only one circle can be passed, and shows how to find the center of that circle.

EXERCISES

NOTE. — Concerning the drawing of the figures for these exercises see Sec. 137.

1. To find a point on a given straight line equidistant from two given points on the same side of it.

2. To find a point on a given straight line equidistant from two given points, one on either side of it.

3. To find the points of a given circle that are equidistant from two given points.

4. To find the points of a given circle that are at a given distance from a given point.

5. In a given circle, to lay off a chord of a given length.

6. To find the point on one side of a given acute angle from which a perpendicular of given length can be drawn to the other.

7. To find a point that is equidistant from the sides of a given angle, and 4 in. distant from a given point P.

Suggestion. Considering first the condition that the point is to be equidistant from the sides of the angle, we know that the locus of all such points is the bisector of the angle. Consequently, the desired point is a point of the bisector. Considering next the condition that the point is to be 4 in. distant from *P*, we know that the locus of all such points is a circle of radius 4 in. about *P* as a center. Consequently, the desired point is a point of this circle.

8. Find a point equidistant from the three sides of a triangle. That is, prove Prop., Sec. 177, 1 by means of loci.

9. Find a point within a given angle 3 in. distant from one side and $1\frac{1}{2}$ in. from the other.

10. Find a point that is located 2 in. from each side of a given angle.

11. Draw a circle of given radius, and passing through two given points.

12. Find the centers of all circles of given radius that pass through two points.

13. Find the points that are at a given distance d from each of two given points.

14. Find a point at a given distance a from a given point and a given line.

15. Find a point that is equidistant from two given points and at a given distance from a third point.

16. Determine a point of a given straight line that is at a given distance from another (non-parallel) straight line.

17. Find a point that is equidistant from two parallels and a transversal.

18. Find a point that is equidistant from one pair of consecutive vertices of a quadrilateral, and also equidistant from the other pair of vertices. Solve the same problem for opposite vertices.

182. Methods of Proof. We summarize here some of the most important methods used in the proofs of elementary geometry.

I. **The Synthetic Method.** The putting together of known truths in order to obtain a new truth is called the **synthetic method.**

We have generally used this method in stating the proofs of the preceding theorems, and it is thus sufficiently illustrated. It is the method ordinarily used for the permanent record of finished proofs. Its advantages for this purpose are obvious. But it is subject to the serious objection of seldom showing *why* the particular truths used are put together rather than some other ones. The truth of the steps themselves is clear, but it is frequently not clear what reason there is to expect in advance that these particular steps will lead to the desired result; that is, the synthetic presentation often gives little inkling of how the proof was or might have been discovered. For the discovery of proofs and the solution of exercises, the analytic method, which we are about to describe, is much more promising.

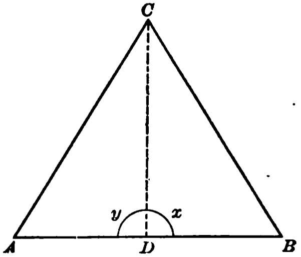
II. **The Analytic Method.** The analytic method pulls apart the statement to be proved into its constituent parts and proves them. It asks what is needed to prove the given statement, and then asks how what is needed can be proved. An example will aid in understanding the nature of the method.

EXAMPLE

The median of an isosceles triangle is perpendicular to the base.

Given $AC = CB$; $AD = DB$.

To show $CD \perp AB$.



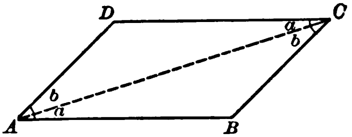
1. If we can show that $\angle x = \angle y$, the proposition will be proved. There are various ways of proving two angles equal. One way is to show that they are corresponding angles of congruent triangles.

2. Hence, if we can show that ▲ ADC and CDB are congruent, the proposition will be proved.

3. We examine what is given to see whether it can be utilized to prove these ▲ congruent. We have $AC = BC$, and $AD = DB$ given; further CD is common to the two triangles. These facts are sufficient to prove the triangles congruent (Sec. 107) and the theorem is proved.

The analytic method is the most powerful method for the discoverer. Recourse should be had to it in the solution of exercises whenever a synthetic proof is not readily found. To acquire skill in the use of the analytic method it would be an excellent exercise to present analytically some of the proofs given in the text. The following example will illustrate what is meant.

The opposite sides of a parallelogram are equal, and the opposite angles are equal.



Given that $ABCD$ is a parallelogram.

To prove $AB = DC$, $AD = BC$, $\angle A = \angle C$ and $\angle B = \angle D$.

We can prove lines and angles equal by showing that they are corresponding parts of congruent triangles. But the figure involves no triangles. So we draw the diagonal AC with the purpose of forming triangles.

If we can show that the two triangles ACD and ABC so formed are congruent, we shall have at once that $AD = BC$, $AB = CD$ and $\angle B = \angle D$.

We have various methods of proving two triangles congruent. To see whether any of these methods serve here, we count up what we know about equality of parts of these triangles. First, line AC is common. The only thing further that we know directly is that $ABCD$ is a ▱. We wish to find equal lines or angles in the figure. The fact that $DC \parallel AB$ will help us. For we know that the transversal AC makes the angles marked a equal, and those marked b equal. Hence in the triangles we have two angles and the included side of the one equal to two angles and the included side of the other. This proves the congruence of the triangles, and therewith all of the proposition except $\angle A = \angle C$. That this is true is apparent from the figure; since each angle equals $\angle a + \angle b$.

EXERCISES

1. Prove that the lines bisecting the base angles of an isosceles triangle, and included within the triangle, are equal.

2. Prove that if the middle points of the sides of a triangle are joined by straight lines, the four triangles thus formed will be equal and each will have the same angles as the given triangle.

3. Prove that a line drawn perpendicular to either side of a right triangle at its middle point passes through the middle point of the hypotenuse.

4. Prove that the sum of the perpendiculars dropped from the middle points of two sides of a triangle to the third side is equal to the altitude on the third side.

5. The bisectors of the equal angles of an isosceles triangle meet the equal sides at points A and B; prove that AB is parallel to the base of the triangle.

6. Prove that the perpendiculars dropped from the middle points of two sides of a triangle to the third are equal.

7. Prove that in an isosceles triangle the vertical angle is twice the angle formed by the base and the altitude on one of the legs.

8. Prove that if two adjacent sides of a quadrilateral are equal and the other sides are equal the diagonals of the quadrilateral intersect at right angles.

9. Prove that if from a point in the base of an isosceles triangle parallels to the legs are drawn, the perimeter of the parallelogram formed equals the sum of the legs of the triangle.

10. Prove that the bisectors of two adjacent angles of a parallelogram are perpendicular to each other.

11. Prove that the diagonals of a rhombus are unequal.

12. How many sides has the polygon the sum of whose interior angles is twice the sum of its exterior angles made by producing each of the sides in succession?

13. Prove that if the sides of any quadrilateral are bisected, the figure formed by joining the adjacent points of bisection is a parallelogram.

14. In the parallelogram $ABCD$, E and F are the middle points of the sides BC and AD respectively; prove that the lines AE and CF trisect the diagonal BD.

15. Prove that the lines joining in order the middle points of the sides of any quadrilateral form a parallelogram whose perimeter is equal to the sum of the diagonals of the quadrilateral.

16. Prove that if two adjacent angles of a quadrilateral are right angles, the bisectors of the other angles are perpendicular to each other.

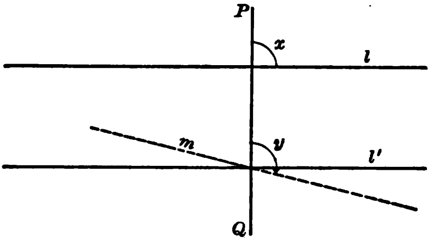
17. Find the number of sides of a regular polygon if (a) an interior angle is 135°, (b) an exterior angle is 72°. Give the name of the polygon in each case.

18. Prove that if one of the acute angles of a right triangle is 60°, the hypotenuse is double the shorter side.

III. The Indirect Method. The indirect method, or the method of ***reductio ad absurdum***, assumes the given proposition to be false, and then proves that this assumption is incorrect.

EXAMPLE

If a line is perpendicular to one of two parallels, it is perpendicular to the other. (Sec. 134.)



In the figure assume that $PQ \perp l$, but not perpendicular to l'.

Then, some other line, as m, is perpendicular to l'.

Hence, $\angle x = \angle y$ and $m \parallel l$, making two lines through one point each parallel to l.

But, this is impossible (Sec. 127); hence, it is absurd to suppose that PQ is not perpendicular to l', and the proposition is proved.

This method has already been used in proving several propositions, but it is of less frequent use than the synthetic or the analytic method.

EXERCISES

Use any of the above methods:

1. Prove that if the median of a triangle is equal to half the side to which it is drawn, the triangle is a right triangle.

2. Prove that in a rectangle not a square the bisectors of the angles inclose a square.

3. The median to the base of an isosceles triangle is perpendicular to the base.

4. The perpendicular bisector of the base of an isosceles triangle passes through the opposite vertex.

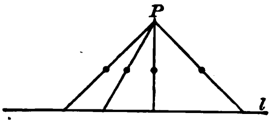
5. Perpendiculars are drawn from a point within an acute angle to the sides. Prove that the angle between the perpendiculars is an obtuse angle.

6. Prove that a triangle can have but one obtuse angle.

7. What is the greatest number of obtuse angles possible among the angles of a quadrilateral?

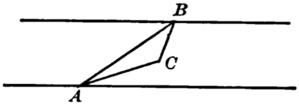
IV. **Method of Loci.** If points satisfying given conditions are sought, the method of loci is used. (Sec. 181.)

EXERCISES



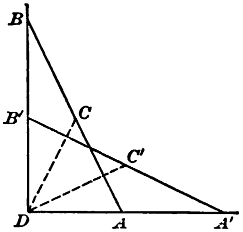
1. Find a point on one side of a triangle that is equidistant from the other two sides.

2. What is the locus of the middle points of all lines drawn from a given point to a given straight line?



3. A triangle glides so that two of its vertices remain on two given parallels. What is the locus of the third vertex?

4. A straight line moves so that its extremities always rest on the two sides of a right angle. What is the locus of its mid-point?



Suggestion. 1. Assume a perpendicular from C to DA. Then $DC = CA$. Secs. 165 and 117, II.

Furthermore, CA is always the same.

5. The locus of all triangles having a given segment as base and altitude of given length consists of two straight lines parallel to the line of the base, and at a distance from it equal to the given altitude.

6. The locus of the third vertex of all isosceles triangles on a given segment as base is the perpendicular bisector of the base.

183. Suggestions for proving propositions:

1. *Draw accurate figures.*

It is easier to discover the relations that lead to a proof if the figure is substantially correct. Thus, if an angle is given equal to another, it should be drawn approximately so.

2. *Avoid special figures.*

Examples: If it is required to prove that an exterior angle of a triangle is equal to the sum of the opposite interior angles, it would not be sufficient to take an equilateral triangle and use the fact that each angle is 60°, for this would not prove the proposition for the more general case of a scalene triangle. Again, if the proposition relates to a parallelogram, a rectangle should not be taken.

3. *State and locate in the figure exactly what is given and what is to be proved.*

4. *Apply the method of loci when the exercise calls for loci or the method seems promising for some other reason; otherwise, unless a synthetic proof is easily found, try the analytic method; lastly, try the indirect method.*

5. *Seek to apply standard propositions that treat of the given relations:*

Examples:

(*a*) If two lines are to be proved equal, try to prove that they are homologous sides of congruent triangles, or sides of the same equilateral triangle, or opposite sides of a parallelogram.

(*b*) If two angles are to be proved equal, try to prove that they are corresponding angles of congruent triangles, or base angles of an isosceles triangle, or corresponding or alternate angles made by a transversal of parallels, or opposite angles of a parallelogram, or that they are both complements of the same angle, or both supplements of the same angle, or that they have their sides parallel right side to right, and left to left, or that they have their sides perpendicular, right to right and left to left.

(*c*) If one angle is to be proved greater than another, try to prove that the angles are opposite to the unequal sides of a triangle, or that one is an exterior and the other an opposite interior angle of a triangle.

(*d*) If one line is to be proved longer than another, try to prove that the lines are opposite to the unequal angles of a triangle.

(*e*) If two triangles are to be proved congruent, try to prove that two sides and the included angle of one are equal respectively to two sides and the included angle of the other, or that a side and two angles of one are equal respectively to a side and any two angles of the other, or that the three sides of one are equal respectively to the three sides of the other.

(*f*) If two lines are to be proved parallel, try to prove that one of the relations of Prop. xxi, p. 68, holds, or that one of the relations of Ex. 16, p. 71, holds, or that each given line is parallel to a third line, or that two points of one given line are equidistant from the other given line.

NOTE. It is often necessary to assume auxiliary lines in order to apply these suggestions.

HISTORICAL NOTE

PLATO

The various styles of proof or methods of procedure used in seeking the truths of geometry are, in the main, not modern. It is an interesting fact that the best of them are as old as geometry itself. The analytic method, perhaps the most helpful of all, is due to Plato; at least this style of proof was first formulated and brought into prominence by the Platonic school.

PLATO

The early philosophers associated the study of mathematics closely with the study of philosophy, but many of them used a minimum amount of mathematical truth on which to base a large amount of speculative doctrine. But Plato, who deplored the shallowness of the Sophists, recommended to his pupils the attainment of a thorough mastery of mathematics. He even enforced this as a preliminary requirement by placing over the entrance to his school the warning: "Let no one unacquainted with geometry enter here." So greatly impressed was he with the evidence of mathematical law operating in the physical world, that he referred to the Creator as "continually geometrizing."

Plato was born at Athens (429 B.C.), studied under Socrates, traveled in Egypt, Italy and Sicily, and was at one time a pupil of the mathematician Theodorus of Cyrene. On returning to his native city, Plato founded a school of philosophy in the groves of the *Academia*, which was known as The Academy, and there, by reason of the rigorous requirements in mental training, produced a large number of mathematicians. The Academy contributed many of the accurate definitions of geometry and some of the axioms. It is responsible for the peculiar limitation that the only instruments admitted shall be the compasses and the unmarked straightedge. The followers of Pythagoras often indulged in literary fiction; for example, a point was defined by them to be "unity in space," but the Platonists defined a point as the beginning or boundary of a line, and to them is due the useful conception that the point, line and surface are boundaries of lines, surfaces and solids, respectively. Plato is said to have solved the Delian Problem; that is, to construct a cube double the volume of the cubical altar of Apollo. He cannot, however, have done this by elementary geometry, for like the other two famous problems of antiquity, "the trisection of the angle" and "the squaring of the circle," this problem has since been shown to be unsolvable by means of the straightedge and compasses.

BOOK II

CIRCLES

184. Preparatory.

1. Review the definitions of:

Circle (Sec. 67).	Diameter (Sec. 70).
Circumference (Sec. 74).	Arc (Sec. 71).
Radius (Sec. 69).	Chord (Sec. 72).

2. Of what points is the circle the locus? (Sec. 180, 3.)

3. How do all radii of the same circle compare in length?

4. Compare the length of a diameter of any circle with the length of its radius.

5. How do all diameters of the same circle compare in length?

ELEMENTARY PROPERTIES

185. Equal Circles. Two circles that can be made to coincide are **equal.**

186. 1. *Two circles that have the same center and equal radii coincide.*

2. *Two circles that have the same center and one point in common coincide.*

A circle can be drawn when we know its center and one point.

If the diameter is given, the center of the circle will be the mid-point of the diameter.

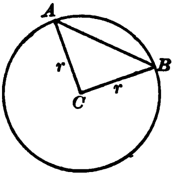
3. *Two circles that have equal radii are equal, and conversely.*

187. *The centers of all circles that can be passed through two given points are on the perpendicular bisector of the line joining them.* (Sec. 181, 6.)

188. *Through three non-collinear points, one and only one circle can be passed.*

For, there is only one point equidistant from the three points (Sec. 177, 2), and this point, together with its distance from each of the given points, determines one and only one circle.

189. *The diameter is the longest chord of a given circle.*



For, considering any chord, AB, not a diameter, we have:

$$AB < AC + CB, \text{ or,}$$
$$AB < 2\,r.$$

But, $$2\,r = \text{Diameter.}$$
$$\therefore AB < \text{Diameter.}$$

190. The chord of an arc is said to **subtend** the arc.

191. Semicircle. The arc subtended by a diameter is called a **semicircle.**

The term semicircle is sometimes used to designate the portion of the plane inclosed by a diameter and one of its subtended arcs.

192. Equal Arcs. Two arcs that can be made to coincide are **equal.**

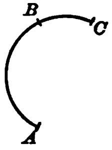
193. *The two arcs subtended by a diameter are equal; that is, the diameter bisects the circle.*

For, by folding along the diameter, the two arcs can be made to coincide.

194. *Two arcs of the same circle, or of equal circles, can be so placed that two end points coincide, and that the arcs are in complete contact throughout the entire extent of one of them.*

195. Sum of Two Arcs. Two arcs of the same circle, each less than a semicircle, placed end to end on the circle so as not to overlap, form an arc called their **sum**.

In the figure, $AC = AB + BC.$



If one, or both, of the arcs are equal to or greater than a semicircle, they may extend sufficiently to meet or to overlap. In this case the sum is either a circumference or more than a circumference.

Thus, in Fig. 1, the sum of arcs ACB and BDA is a circumference; in Fig. 2, the sum of arcs $AHEF$ and FAH is more than a circumference.

In what follows, we shall always understand that *the arcs are not greater than a semicircle,* unless otherwise specified.

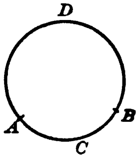


FIG. 1

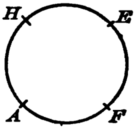


FIG. 2

196. Difference of Two Arcs. When two arcs are placed as in Sec. 195 and one of them projects beyond the other, the projecting portion is the **difference** of the two arcs.

In Fig. 2, $HE = FEH - FE.$

197. Intercepted Arcs. The arc connecting the end points of two radii is said to be **intercepted** by the angle at the center.

When there is no danger of confusion, a circle is sometimes referred to by its center only. Thus, $\odot C$ means "the circle whose center is at C."

198. *Every chord not a diameter subtends one arc less than a semicircle, and one greater than a semicircle.*

In speaking of the arc subtended by a chord, we shall mean the *arc less than a semicircle* unless otherwise stated.

199. *If two arcs of the same or equal circles are equal, their chords are equal and if the chords are equal, the arcs are equal.*

For, they can be made to coincide.

EXERCISES

1. Mark a point. Draw a curve that shall contain all points at distance 2 in. from the given point.

2. Draw an angle. Draw the arc that contains all points between the sides of the angle at distance $2\frac{1}{2}$ in. from the vertex of the angle.

3. Draw a triangle having one side 3 in. long, and the others of any convenient length. Mark a point *P* within the triangle. Draw the curve that contains all points 2 in. distant from *P*. Indicate all points on the sides of the triangle that are 2 in. distant from *P*. (The number of such points may differ in different figures.)

4. Draw a circle of radius 2 in. Mark a point *P* within it, other than the center. Draw an arc through all points within the circle that are 2 in. distant from *P*. Indicate the points of the circle that are 2 in. from *P*.

200. Preparatory.

1. Read Theorem IV, Book I, p. 38; also Theorem V, p. 40.
2. What is given in Theorem IV? What is to be proved in Theorem V?
3. What is given in Theorem V? What is to be proved in Theorem IV?

201. Converse Theorems. The theorem which results from interchanging the hypothesis and the conclusion of a given theorem is called the **converse** of the given theorem.

For example, Proposition XV, p. 56, is the converse of Proposition XVI, p. 57, and *vice versa.*

The converse of a theorem may or may not be true; hence any assertion concerning its truth requires proof.

For example: If a figure is a square, it has four sides. The converse is: If a figure has four sides, it is a square (which is not necessarily true).

EXERCISES

What is the converse of each of the following statements? Is the converse true? If so, why?

1. If a closed figure has three sides, the figure is a triangle.

2. If $4x = 8a$, then $x = 2a$.

3. If $x = 3$, then $x^2 = 9$.

4. If $a > b$, then $c > d$.

5. State the converse of: If a triangle has two equal sides, the angles opposite these sides are equal. Where has this converse been proved?

6. If a triangle is equilateral, it is also equiangular. State the converse and tell where it has been proved.

7. A point on the perpendicular bisector of a line segment is equidistant from the ends of the segment. State and prove the converse.

8. If two triangles have two sides of the one equal to the corresponding sides of the other and the included angles unequal, the third sides are unequal. What is the converse?

202. A special case in which the converse is always true. If four quantities, a, b, c, d, are so related that

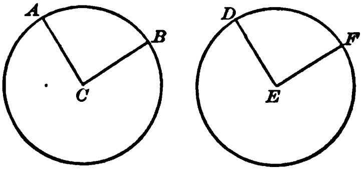
(1) if $a > b$ then $c > d$,
(2) if $a = b$ then $c = d$,
and (3) if $a < b$ then $c < d$,

it follows that the converses of these three relations are true.

For, when $c > d$, a cannot equal b, since then $c = d$ by (2); nor can a be less than b, for then $c < d$ by (3); hence, if $c > d$, then $a > b$. In the same way the converses of (2) and (3) can be proved.

PROPOSITION I. THEOREM

203. *In the same circle or in equal circles, equal angles at the center intercept equal arcs, and conversely.*



Given $\odot C = \odot E$, and $\angle ACB = \angle DEF$.

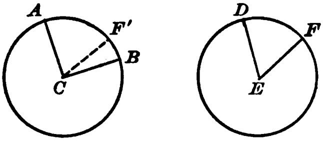
To prove that arc $AB =$ arc DF.

Proof. 1. Make $\angle ACB$ coincide with $\angle DEF$. Sec. 80, VII

2. Then, A coincides with D, and B with F, and arc AB with arc DF. Sec. 67

3. Then arc $AB =$ arc DF. Sec. 194. Q.E.D.

204. COR. 1. *If two central angles are unequal, the larger angle subtends the longer arc, and conversely.*



For, place $\angle DEF$ on $\angle ACB$ so that DE coincides with AC, and DF falls along AB.

Then, EF will fall at CF', since $\angle ABC > \angle DEF$.

Therefore, arc $AB >$ arc DF.
Secs. 196, 194

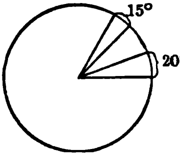
Conversely, if arc $AB >$ arc DF and if arc DF is placed upon arc AB as above, then $\angle ACB$ extends beyond $\angle ACF'$, which is equal to $\angle DEF$.

Hence, $\angle ACB$ is the greater angle.

205. COR. 2. *Equal arcs subtend equal chords.*

Suggestion. Place the arcs and subtended angles upon each other as in Cor. 1, or apply Sec. 199.

206. Measurement of Arcs. Since equal arcs subtend equal angles at the center of a circle, the units of measure, degrees, minutes and seconds, are applied to arcs as well as to the angles which they subtend.

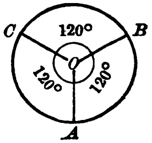


Thus, an arc of 20° means an arc subtending a central angle of 20°. An angle is said to be **measured by an arc** if they both contain the same number of degrees.

EXERCISES

1. Read the number of degrees in arc AB; in BC; in CA.

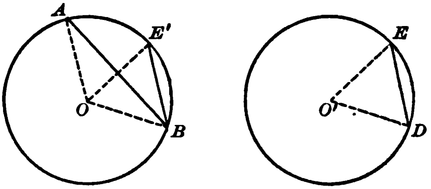
2. If four points, A, B, C, D, divide a circle into four equal arcs and the radii, OA, OB, OC, OD, are drawn, name the size of the angles AOB, BOC, COD and DOA.



3. In the same way, consider six points, A, B, C, D, E, F.

Proposition II. Theorem

207. *In the same circle or in equal circles, the greater of two arcs is subtended by the greater chord, and conversely.*



Given arc $AB >$ arc ED.

To prove that chord $AB >$ chord DE.

Proof. 1. Place arc DE on arc AB as at BE'. Sec. 196

2. $\angle AOB > \angle E'OB$. Sec. 204

3. Draw the radii to the end points of the chords.

4. In $\triangle AOB$, $E'OB$, side $AB >$ side $E'B$, or ED. Sec. 121

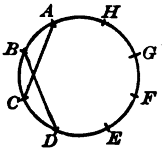
5. $\therefore$ chord $AB >$ chord DE. Q.E.D.

The proof of the converse is left as an exercise.

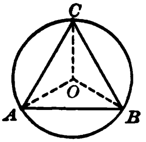
NOTE. — In step 4, the restriction is implied that neither arc shall exceed a semicircle. Otherwise, the proposition is not necessarily true.

208. COROLLARY. *Equal chords subtend equal arcs.*

EXERCISES



1. The points $A \cdots H$ divide the circle into eight equal arcs. Prove that chord $AC =$ chord BD; and that all chords which join any point of division to the next but one in either direction are equal. Likewise prove that chord $AD =$ chord HE.

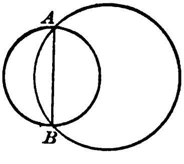


2. In the figure $\triangle ABC$ is equilateral. How many degrees in $\angle AOB$? In $\angle BOC$? In $\angle COA$?

3. Given two diameters of a circle perpendicular to each other. How many degrees in the arcs formed by adjacent radii of the two diameters?

4. Prove the converse of Proposition II, Sec. 207, by applying Sec. 202 to Secs. 207 and 208.

5. Assume a circle to be drawn through the four corners of a square. Compare the four arcs subtended by the four sides of the square.



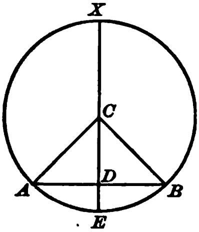
209. Common Chord. If two circles intersect, the line joining their points of intersection is called their **common chord.**

In the figure AB is the common chord.

The line connecting their centers is called the **line of centers.**

PROPOSITION III. THEOREM

210. *The radius perpendicular to a chord bisects it and the arc which it subtends.*



Given chord AB, and CE perpendicular to it.

To prove $AD = DB$, and arc $AE =$ arc EB.

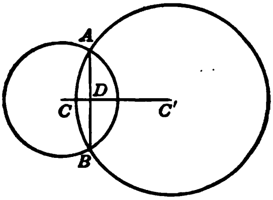
Proof. 1. $\triangle\triangle\, ACD$ and CDB are congruent. Sec. 112

2. Hence, $AD = DB$.

3. Since $\angle ACD = \angle DCB$, arc $AE =$ arc EB.

Sec. 203. Q.E.D.

211. COROLLARY. *The line of centers of two intersecting circles bisects their common chord, and is perpendicular to it.*



Proof. 1. Let AB be the common chord of two circles with centers C and C'. Let D be the mid-point of AB.

2. Connect D with C, also with C'.

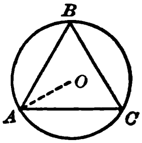
3. Draw CA and CB, then, $DC \perp AB$. Similarly $DC' \perp AB$. Secs. 107, 46

4. $\therefore$ CDC' is a straight angle, and CC' is a straight line. Sec. 38

5. $\therefore$ CC', the line of centers, is $\perp AB$ and bisects it.

EXERCISES

1. In the figure $\triangle ABC$ is equilateral, and O is the center of the circle. Prove that OA bisects $\angle BAC$.



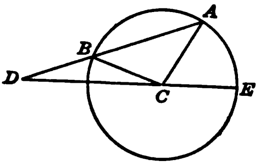
2. The bisector of an angle at the center of a circle bisects the subtended chord.

3. If a chord AB is produced by the length BD equal to the radius, and the line DC through the center C cuts the circle in E, show that $\angle ACE = 3 \angle ADE$.

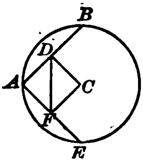
Suggestion. $\triangle ACB$ and $\triangle DBC$ are isosceles. Why?

$\angle ACE = \angle ADE + \angle DAC$

$= \angle ADE + \angle ABC$. Why?

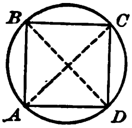


4. Draw chords AB and AD in a circle with center C. Connect C with the mid-point E of the chord AB and with the mid-point F of chord AD. How many degrees in $\angle AFC$? In $\angle AEC$? Show that angles A and C are supplementary.

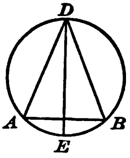


5. CD and CF are perpendicular to AB and AE. If $AB = AE$ show that the triangle ADF is isosceles.

6. Chords AB, BC, CD are equal. Show that chord $AC =$ chord BD.



7. Prove that if the chord AB is perpendicular to the diameter DE, the triangle ADB is isosceles.



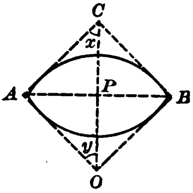
8. If two arcs have the same extremities and contain the same number of degrees, the radii are equal.

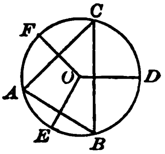
Suggestion. Connect C and O with P, the mid-point of AB.

CPO is a straight line. Why?

$\angle C = \angle O$. Given

Then, $\angle x = \angle y$. Why?

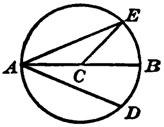




9. In the figure OD, OE and OF bisect the sides of the triangle ABC. Name three pairs of equal arcs.

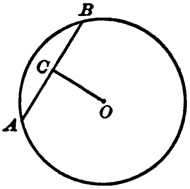
10. Given that AB is a diameter of the circle with center at C, and arc DB = arc BE. Prove that

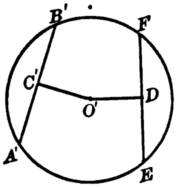
(*a*) Chord BE = chord BD.
(*b*) Arc AE = arc AD.
(*c*) Chord AE = chord AD.



PROPOSITION IV. THEOREM

212. *In the same circle or in equal circles, equal chords are equally distant from the center, and conversely.*





Given the equal circles O and O', with chord AB = chord EF.

To prove that the perpendiculars OC and $O'D$ are equal.

Proof. 1. Make the circles coincide; then ACB takes the position $A'C'B'$.

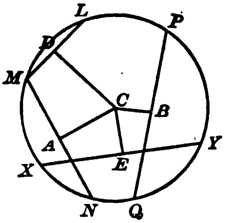
2. Turn circle O on circle O' until arc $A'B'$ coincides with arc FE. Then, chords $A'B'$ and FE coincide. Secs. 194, 205

3. Then the two perpendiculars also coincide.

4. Hence, the chords are equidistant from the center. Q.E.D.

The proof of the converse is left as an exercise.

EXERCISES

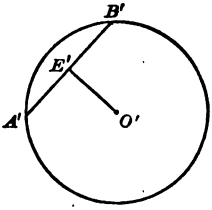


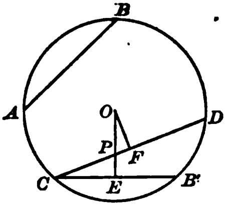
1. In the figure the perpendiculars satisfy the relation $CD > CA > CE > CB$. Arrange the chords in order of increasing magnitude.

2. Show that if two chords intersect and make equal angles with the diameter through this common point, the chords are equal.

Proposition V. Theorem

213. *In the same circle or in equal circles, the shorter of two unequal chords is at the greater distance from the center, and conversely.*





Given two equal circles and chord $CD >$ chord $A'B'$ or AB. Also OF and $O'E'$ the distances of the chords from the center.

To prove that $OF < O'E'$ or OE.

Proof. 1. Place the circles so that the chords have an end in common, and so that the arc $A'B'$ or AB is in contact with the arc CD, and takes the position CB'. Then $O'E' = OE$.

2. Arc $CD >$ arc CB' and extends beyond it. Sec. 207

3. Then, chord CD will lie between O and chord CB'. Hence, the line OE will cut CD at a point P other than F.

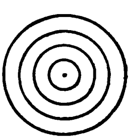
4. Then, $OE > OP$. Why?

5. Also, $OP > OF$. Why?

6. $\therefore OE > OF$ or $OE > O'E'$. 78, VIII. Q.E.D.

Conversely, if chord AB is farther from the center than chord CD, then chord AB < chord CD.

For if chord AB is *not* less than chord CD, it is either (1) equal to chord CD, or (2) greater than it. If (1) it is equal to chord CD, then (Sec. 212) chord AB would be at the same distance from the center as chord CD, contrary to hypothesis. If (2) chord $AB >$ chord CD, then (Sec. 213) chord AB is nearer the center than chord CD, contrary to hypothesis. Therefore, chord $AB <$ chord CD. Sec. 202. Q.E.D.



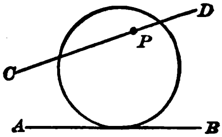
214. Concentric Circles. Two or more distinct circles that have the same center are called **concentric circles.**

215. Intersections of a Straight Line and a Circle.

No straight line can have three distinct points in common with a circle, for three collinear points do not determine a circle.

Accordingly, a straight line may have in common with a circle two distinct points, one point or no point.

216. Secant; Tangent. An unlimited straight line that has two distinct points common with a circle is called a **secant**; one that has but one point common with a circle is called a **tangent** to the circle.



Thus, in the figure, CD is a secant, and AB is a tangent.

A straight line passing through a point P, within the circle, must have two points common with the circle. For the unlimited straight line extends beyond the circle in both directions, and therefore cuts the circle in two points, one in each direction from P.

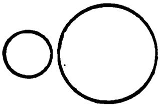
As the tangent has only one point in common with the circle, it can have no point within the circle. The common point of the tangent and the circle is called the *point of contact* of the tangent. Since every other point of the tangent is outside the circle and therefore at a distance of more than the length of the radius from the center, the point of contact is the point of the tangent that is nearest the center.

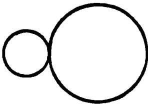
217. Intersections of Two Circles.

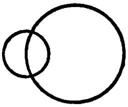
1. *Two different circles can have, at most, two points in common.*

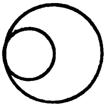
For, if they had three points in common, they could not be different. Sec. 188.

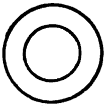
2. *Accordingly, two different circles may have in common two points, one point or no point.*


(1)
No Point


(2)
One Point


(3)
Two Points


(4)
One Point


(5)
No Point

218. Tangent Circles. Two circles that have one and only one point in common are said to be **tangent** to each other.

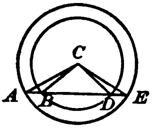
The circles of figures (2) and (4) above are **tangent**. Those of figure (2) are said to be tangent **externally**, those of figure (4), **internally**.

EXERCISES

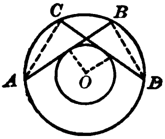
1. If a line cuts two concentric circles, the line segments between the two circles are equal.

2. If two concentric circles are cut by a straight line $ABDE$, prove that $\angle ACB = \angle DCE$.

Suggestion. Prove $\triangle ACB$ congruent to $\triangle CDE$.

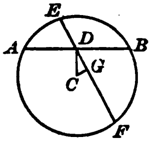


3. If two circles are concentric, prove that all chords of the outer circle that are tangent to the inner circle are equal. Under these conditions, prove that chord $AC =$ chord BD.



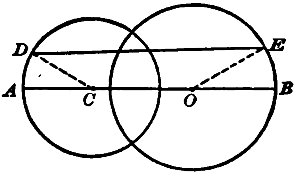
4 Show that all chords of given length in a circle are tangent to a concentric circle.

5. If two chords intersect at the mid-point of one of them and not of the other, prove that the chords are unequal.



Suggestion. Let the chords intersect at D, the mid-point of AB. Then, $CD \perp AB$, but not to EF. Let $CG \perp EF$. Compare CG with CD. Then compare EF with AB.

6. For any two circles, show that the longest distance from a point of one to a point of the other lies in the line of centers.



Suggestion. To show $AB > DE$.

$$\begin{aligned} AB &= AC + CO + OB \\ &= DC + CO + OE, \end{aligned}$$

which is greater than what line?

The proof holds if the circles are tangent or lie without or within each other. Make figures for these cases, varying also the position of DE, and repeat the proof.

7. The shortest distance between two circles that do not intersect is on their line of centers.

8. If the end points of equal chords of equal circles are joined to the corresponding centers, congruent isosceles triangles are formed.

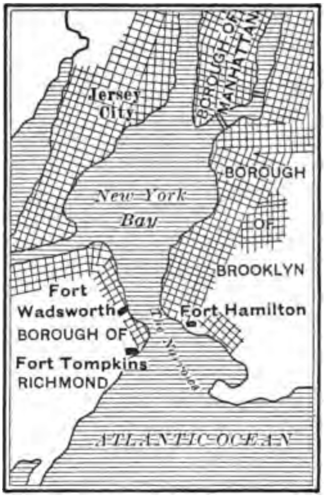
9. The guns of a certain fort have a range of 3.5 mi. in all directions. Using 1 in. to represent a mile, draw a diagram to show the area in which an enemy would be exposed to the fire of the guns.

10. Forts Hamilton (H) and Wadsworth (W), protecting New York Harbor, are 1 mi. apart. Some of the guns of H

have a range of $4\frac{1}{2}$ mi., those of W $5\frac{1}{2}$ mi. Using $\frac{1}{2}$ in. to represent 1 mi., make a diagram to show the area covered by the guns of each fort.

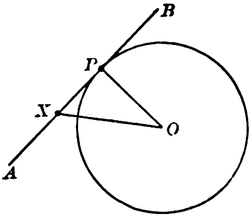
11. Shade or color differently the areas covered by the guns of W alone; by those of H alone; by those of both forts.

12. A square lawn of side 100 ft. is watered by a hose which can throw water 60 ft. Draw different figures showing the part that can be watered from one corner; from the middle of a side; from the center of the square.



PROPOSITION VI. THEOREM

219. *The radius to the point of contact of a tangent to a circle is perpendicular to the tangent.*



Given AB tangent to circle O, and OP the radius to the point of contact.

To prove that $OP \perp AB$.

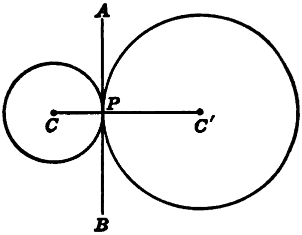
Proof. 1. Let X be any point other than P on AB.

2. Then $OX > OP$. Sec. 216

3. $\therefore OP \perp AB$. Secs. 108, 117, I. Q.E.D.

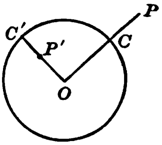
220. Cor. 1. *A line perpendicular to a radius of a circle at its outer extremity is a tangent to the circle.*

221. Cor. 2. *The line of centers of two tangent circles passes through their point of tangency.*



Suggestion. Let AB be the common tangent of the given circles, with the point of contact at P. Connect P with C and with C'. Show that PC and PC' are both perpendicular to AB and hence that CPC' is a straight line. Make a figure and proof for the case of internal contact.

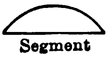
222. Distance of a Point from a Circle. The distance of a point from a circle is its distance from the curve measured on the line determined by the given point and the center of the circle. The line may be produced, if necessary.

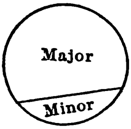


Thus, PC is the distance of point P from the circle, and $P'C'$ is the distance of point P' from the circle.

223. Segment. The figure formed by an arc and its chord is called a **segment** of a circle.

A chord divides a circle into two segments. The smaller is called the **minor segment**, and the larger, the **major segment**.





224. Inscribed Circle. If any figure has all its sides tangent to a circle, the circle is said to be **inscribed** in the figure. If the figure is closed, it is said to be **circumscribed** about the circle.

Thus, we may have a circle inscribed in an angle, as in (Fig. 1), p. 125, or in a triangle as in (Fig. 2), or in a quadrilateral (Fig. 3), or in a pentagon (Fig. 4), etc. The triangle, quadrilateral and pentagon are circumscribed about the given circles.

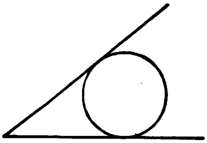


FIG. 1

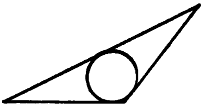


FIG. 2

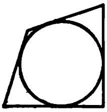


FIG. 3

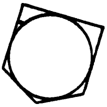
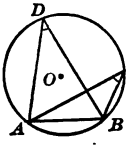


FIG. 4

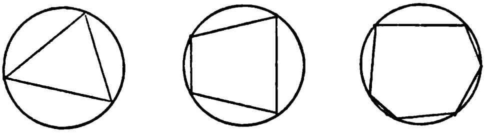
225. Inscribed Angle. An angle with its vertex on a circle and sides cutting the circle again, is said to be inscribed in the circle.

The angle is also said to be inscribed in the arc through the vertex, and terminated by the sides of the angle. It is also said to be inscribed in the segment formed by this arc and its chord.

Thus $\angle ADB$ and $\angle ACB$ are inscribed in the arc $ADCB$ and both are inscribed angles of circle O.



226. Inscribed Figures. If any figure has all its vertices on a circle, it is said to be **inscribed** in the circle. The circle is said to be **circumscribed** about the figure.



The figures show an inscribed triangle, quadrilateral and hexagon.

EXERCISES

1. Show that the angle between two tangents to a circle and the angle between the radii to their points of contact are supplementary.

2. Show that tangents at the end points of a diameter are parallel.

3. What is the locus of the centers of all circles of given radius tangent to a given straight line?

4. What is the locus of the centers of all circles tangent to a given straight line at a given point?

5. Two points of a tangent equidistant from its point of contact are equidistant from the circle.

6. The point of a circle with center *C* that is nearest to an exterior point *P* is the point where the line *PC* cuts the circle.

7. What is the locus of the centers of all circles of given radius that are tangent to a given circle externally? Also internally?

8. What is the locus of the centers of all circles of given radius and passing through a given point?

9. What is the locus of the centers of all circles of given radius that are tangent to a given square externally? Internally?

10. What is the locus of the centers of all circles that are tangent to two given intersecting straight lines?

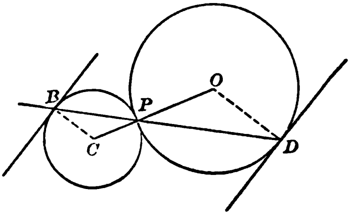
11. What is the locus of the centers of all circles tangent to two given parallel lines?

12. If two circles are tangent, a tangent to one of them at their point of contact is a tangent to the other.

13. Two tangent circles can have one, and only one, common tangent at their point of contact.

14. If two circles are tangent, their point of contact and the points of contact of any two parallel tangents to the two circles lie in the same straight line.

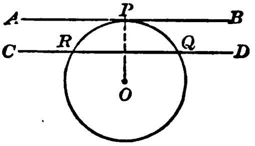
Suggestion. Join *PB* and *PD*. *CO* passes through *P*, Sec. 221. Prove *BPD* is a straight line. To do this, prove $\angle CPB = \angle DPO$, by observing that $OD \parallel BC$. Why? Also, $\angle BCP = \angle DOP$. Also, $\triangle DOP$ and $\triangle BCP$ are isosceles.

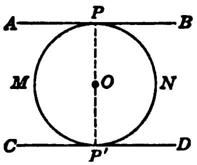


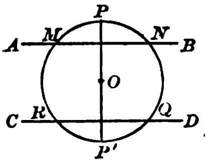
15. Make the figure and prove the above exercise also for the case where the circles are tangent internally.

PROPOSITION VII. THEOREM

227. *Parallel lines intercept equal arcs on a circle.*







Given a circle with center O on which the parallel lines AB and CD intercept arcs shown in the figures.

CASE I. **To prove** arc $PR =$ arc PQ.

1. Given $OP \perp AB$ at the point of tangency. Sec. 219
2. $OP \perp CD$. Sec. 134
3. Arc $PR =$ arc PQ. Why? Q.E.D.

CASE II. **To prove** arc $PMP' =$ arc PNP'.

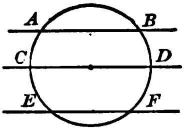
1. Given radius $OP \perp AB$, and radius $OP' \perp CD$. Sec. 219
2. Extend PO to CD. It must coincide with OP'. Why?
3. Then, POP' is a diameter.
4. $\therefore$ arc $PMP' =$ arc PNP'. Why? Q.E.D.

CASE III. **To prove** arc $MR =$ arc NQ.

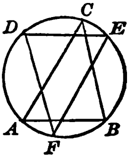
1. POP' is a diameter $\perp AB$ as in Case II. Then, arc PM $=$ arc PN and arc $P'R =$ arc $P'Q$.
2. Arc $PNQP' -$ arc $PN -$ arc $P'Q =$ arc NQ.
3. Arc $PMRP' -$ arc $PM -$ arc $P'R =$ arc MR.
4. $\therefore$ arc $NQ =$ arc MR. Why? Q.E.D.

Prove Case III also when AB and CD are both on the same side of O.

EXERCISES



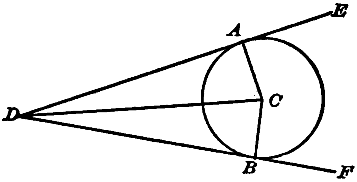
1. Given that the three straight lines are parallel, name three pairs of equal arcs.



2. The sides of $\triangle ABC$ are respectively parallel to those of $\triangle DEF$. Name three pairs of equal arcs.

PROPOSITION VIII. THEOREM

228. *If two tangents to the same circle intersect, the distances from their point of intersection to their points of contact are equal.*



Given DE and DF, tangents to the circle with center C.

To prove that $DA = DB$.

Proof. 1. $\triangle ACD$ is congruent to $\triangle BCD$. Why?

2. $\therefore DA = DB$. Q.E.D.

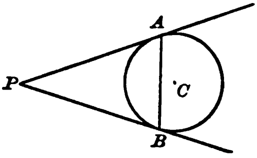
This theorem is sometimes stated briefly thus: *The tangents from a point to a circle are equal.*

229. COROLLARY. *The line from the intersection of a pair of tangents to the center of the circle bisects the angle between the tangents.*

For, in the above figure CD bisects $\angle ADB$. Why?

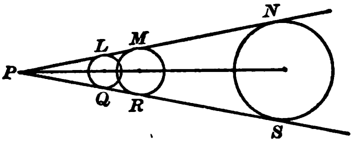
230. Chord of Contact. The line joining the points of contact of two lines tangent to a circle is called the **chord of contact** of those lines.

In the figure AB is the chord of contact of tangents PA and PB.



EXERCISES

1. The lines PN and PS being tangent to the three circles, name six pairs of equal line segments.



2. Prove that the chord of contact of two intersecting tangents is perpendicular to the bisector of the angle between the tangents.

3. If a circle is inscribed in an isosceles triangle, prove that the chord of contact of the equal sides is parallel to the base.

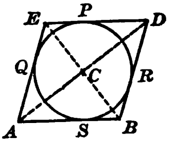
4. If a circle is inscribed in an equilateral triangle, and the points of contact of the sides are joined in pairs, prove that the triangle so found is equilateral.

5. The line from the common point of two intersecting tangents to the center of the circle bisects the angle between the tangents.

6. The diagonals of a parallelogram circumscribed about a circle pass through the center of the circle.

Suggestion. Connect A with C and D with C. To show that ACD is a straight line, show that $\angle E + \angle EDC + \angle EAC =$ a straight angle.

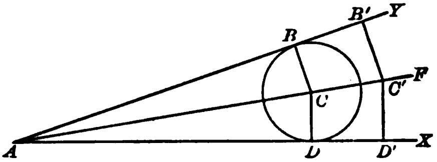
To do this observe that $\angle E + \angle EDB =$ a straight angle, and that $\angle EDC + \angle EAC = \angle EDB$.



7. If a quadrilateral has its four sides tangent to a circle, each pair of opposite sides has the same sum.

PROPOSITION IX. THEOREM

231. *The locus of the centers of all circles inscribed in a given angle is the bisector of the angle.*



Given the angle A and the circle with center C.

To prove that (1) C is on AF, the bisector of $\angle A$; (2) that the circle with any point C' on AF as the center and $C'D'$, the $\perp$ AX, as a radius, is tangent to AX and AY.

Proof of (1). 1. Let CD and CB be the radii to the points of contact of the tangents. Then, the angles at D and B are right angles. Sec. 219

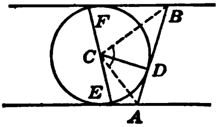
2. $\therefore \triangle ABC$ is congruent to $\triangle ADC$. Sec. 112

3. $\therefore C$ is a point of the bisector of $\angle A$. Q.E.D. Why?

Proof of (2). If C' is any point of the bisector, and $C'D'$ and $C'B'$ are $\perp$ the sides of the angle, then $C'D' = C'B'$ (Sec. 111) and the circle with radius $C'D'$ has AD' and AB' as tangents. Secs. 216, 219. Q.E.D.

EXERCISES

1. If two parallel tangents to a circle are cut by a third tangent, the lines joining the center to the two intersections are at right angles.



Suggestion. Show that $\angle FBA$ and $\angle EAB$ are supplementary and then show that $\angle DAC$ and $\angle DBC$ are complementary angles.

2. Draw two intersecting lines and assume the bisectors of the four angles. What is the locus of the centers of all circles tangent to these two lines?

3. In the figure of Ex. 2, how many circles are there whose centers are at distance *d* from the intersection of the two given lines? How can these centers be found?

4. Show that the intersection of the three bisectors of the angles of a triangle is the center of the circle inscribed in the triangle.

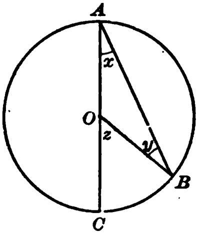
5. The points of contact of the inscribed circle with the sides of a triangle divide the three sides into six segments. Show that the sum of any three non-adjacent segments is equal to one half of the perimeter of the triangle.

6. Prove that the circle inscribed in a square touches each side of the square at its mid-point.

PROPOSITION X. THEOREM

232. *An angle inscribed in a circle is measured by half the intercepted arc.*

CASE I. *One side of the angle passes through the center, and the other side is a secant.*



Given $\angle BAC$ with vertex at A in the circle with center O.

To prove that $\angle BAC = \frac{1}{2} \angle BOC$,

or $\angle x = \frac{1}{2} \angle z$ or $\frac{1}{2}$ arc BC.

Proof. 1. $\angle z = \angle x + \angle y$. Sec. 146

2. $\angle x = \angle y$. Why?

3. $\therefore \angle z = 2 \angle x$.

4. $\therefore \angle x = \frac{1}{2} \angle z$ or $\frac{1}{2}$ arc BC. Why?

CASE II. *Neither side passes through the center.*

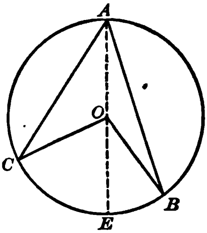


FIG. 1

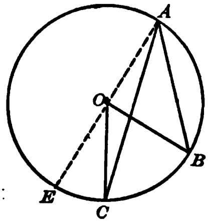


FIG. 2

Given $\angle BAC$ in Fig. 1 and Fig. 2, with vertex A on the circle with center O.

To prove that $\angle BAC = \frac{1}{2} \angle BOC$ or $\frac{1}{2}$ arc BC.

Proof. 1. Draw the diameter AE.

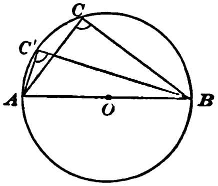
2. $\angle BAE = \frac{1}{2} \angle BOE$ in each figure. Case I

3. $\angle CAE = \frac{1}{2} \angle COE$ in each figure. Case I

4. $\therefore \angle BAC = \frac{1}{2} \angle BOC$, by adding the equations in steps (2) and (3) for Fig. 1, and subtracting equation (3) from (2) for Fig. 2. Q.E.D.

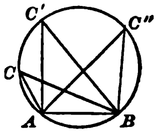
233. COR. 1. *An angle inscribed in a semicircle is a right angle.*

234. COR. 2. *All angles inscribed in the same segment or in equal segments are equal.*

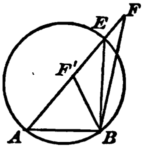


235. COR. 3. *The locus of the vertices of all equal angles that can be erected on a given base and on the same side of it, is an arc of a circle having the given base as chord.*

For, (1) all angles on the given base with their vertices on the arc are equal (Cor. 2).



(2) Any such angle with its vertex not on the arc is not equal to one with its vertex on the arc.



For, if F is without the arc, and if AF cuts the arc at E, then $\angle AEB > \angle AFB$ (Sec. 146). Similarly, if F' is within the arc, $\angle AF'B > \angle AEB$.

Therefore, the arc AEB is the locus of equal angles on base AB. Q.E.D.

EXERCISES

1. The circle described upon the hypotenuse of a right triangle as diameter passes through the vertex of the right angle.

2. Any angle inscribed in a segment less than a semicircle is an obtuse angle.

3. Any angle inscribed in a segment greater than a semicircle is an acute angle.

Throughout the following exercises, C denotes the center of the circle.

In Fig. 1:

4. Name two right angles.

5. Show that BFD is an acute angle.

6. Show that BDE is an obtuse angle.

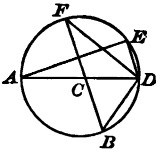


FIG. 1

In Fig. 2:

7. Show that $\angle AFD = \angle EAD$.

8. Name another angle that is equal to $\angle AFB$.

9. Show that arc BD = arc AF, and hence that $\angle BFD = \angle ABF$.

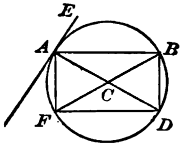


FIG. 2

10. In Fig. 3, arc AB contains 48°. Find the number of degrees in $\angle ADB$; in $\angle A$.

11. In Fig. 3, arc AE contains 100°. Find, in order, the number of degrees in $\angle$s x, y, z.

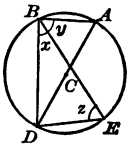
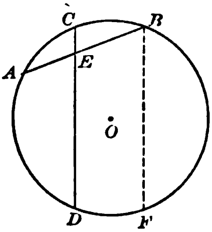


FIG. 3

PROPOSITION XI. THEOREM

236. *An angle formed by two chords intersecting within a circle is measured by half the sum of the arcs intercepted between its sides and the sides of its vertical angle.*



Given AB and CD, two chords intersecting at E within the circle.

To prove that $\angle AED$ is measured by $\frac{1}{2}$ (arc AD + arc CB).

Proof. 1. Through B assume $BF \parallel DC$. Then, $\angle ABF = \angle AED$. Sec. 135

2. $\angle ABF$ is measured by $\frac{1}{2}$ arc AF. Why?

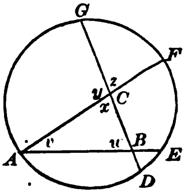
3. Arc AF = arc AD + arc DF.

4. Arc DF = arc BC. Why?

5. $\therefore$ arc AF = arc AD + arc BC.

6. $\therefore \angle AED$ is measured by $\frac{1}{2}$(arc AD + arc CB).
Steps (1), (2) and (5). Q.E.D.

EXERCISES



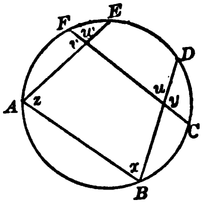
1. In the figure above, prove Proposition XI by drawing the auxiliary line BD instead of the parallel BF.

2. In the figure at the right state the measure of $\angle v$, $\angle w$, $\angle x$, $\angle y$, $\angle z$.

3. Show that in the figure of Exercise 2, $\angle x = \angle z$, and thus prove Prop. I, Bk. I.

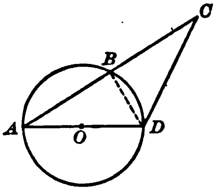
4. In the figure of Exercise 2, find the sum of the measures of angles v, w, x, and thus prove Prop. XXIV, Bk. I.

5. If arc $AB = 150°$, $BC = 50°$, $DC = 40°$, $ED = 60°$, $FE = 25°$, determine the number of degrees in each of the angles u, v, w, x, y, z.

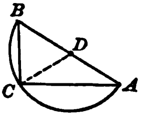


6. A chord AB is extended by its own length, and the end point C joined to the extremity D of the diameter from A. Prove that $DC = AD$.

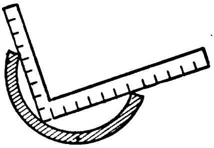
7. In the figure of Exercise 6, show by use of the measuring arcs that $\angle BAD + \angle BDA = 90°$.



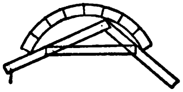
8. D is the mid-point of the hypotenuse of the right triangle ABC. Show that $\angle DCB = \angle DBC$.



9. Explain how a machinist could use a square to test the accuracy of a piece of metal supposed to be bent into the form of a semicircle.

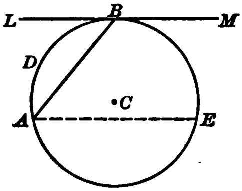


10. To test the accuracy of the workmanship in building an arch in the shape of an arc of a circle, not a semicircle, the architect nailed firmly together three pieces of laths as indicated in the figure. Explain the method of use of this frame.



PROPOSITION XII. THEOREM

237. *An angle formed by a chord and a tangent at one of its extremities is measured by half the intercepted arc.*



Given the chord AB, and line LM, tangent at B.

To prove that $\angle LBA$ is measured by half arc ADB.

Proof. 1. Draw $AE \parallel LM$.

2. Then, $\angle LBA = \angle BAE$. Sec. 135

3. $\angle BAE$ is measured by $\frac{1}{2}$ arc BE. Sec. 232

4. Arc $BE =$ arc ADB. Sec. 227

5. $\angle BAE$ is measured by $\frac{1}{2}$ arc ADB. Steps (3), (4)

6. $\therefore \angle LBA$ is measured by $\frac{1}{2}$ arc ADB.
Steps (2), (5). Q.E.D.

Show that the theorem holds also for $\angle ABM$.

Suggestion. ABM is the supplement of $\angle LBA$.

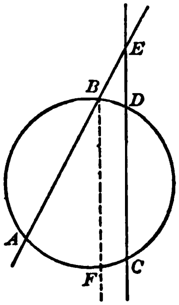
EXERCISES

1. Draw a semicircle of given radius. Take a point P within the semicircle and connect it with the ends of the diameter. Prove that the angle thus formed at P is greater than a right angle.

2. Draw a circle and inscribe any triangle. Assume a tangent to the circle at any vertex of the triangle. Prove that the tangent makes angles with the sides of this triangle at this vertex equal to the non-adjacent angles of the triangle.

PROPOSITION XIII. THEOREM

238. *An angle formed by two secants intersecting without a circle is measured by one half the difference of the intercepted arcs.*



Given AB and CD, two secants intersecting in E.

To prove that $\angle AEC$ is measured by $\frac{1}{2}$ (arc AC − arc BD).

Proof. 1. Through B assume $BF \parallel DC$.

2. Then, $\angle ABF = \angle AEC$. Sec. 135

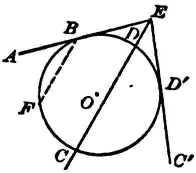
3. $\angle ABF$ is measured by $\frac{1}{2}$ arc AF. Why?

4. Arc AF = arc AC − arc FC = Arc AC − arc BD. Why?

5. $\therefore$ $\angle AEC$ is measured by $\frac{1}{2}$ (arc AC − arc BD).

Steps (2), (3), (4). Q.E.D.

239. COR. 1. *The angle between a secant and a tangent is measured by half the difference of the intercepted arcs.*

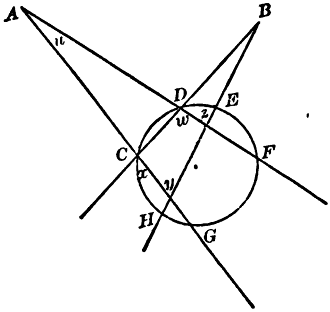


Suggestion. Let EA be the tangent and EC the secant. Draw $BF \parallel EC$.

240. COR. 2. *The angle between two tangents is measured by half the difference of the intercepted arcs.*

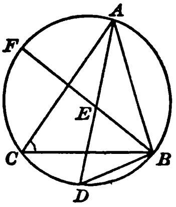
Suggestion. Apply Cor. 1 to $\triangle$ BEC and CED'.

EXERCISES



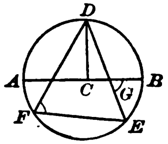
1. Given that arcs $CD = 40°$, $DE = 30°$, $EF = 50°$, $GF = 80°$, $HG = 20°$, find the number of degrees in angles A, B, w, x, y, z.

Suggestion. Find the length of arc CH and apply Secs. 232, 236, 238.



2. If $\triangle ABC$ is inscribed in a circle, and if the bisectors of angles A and B intersect at E, show that $ED =$ chord DB.

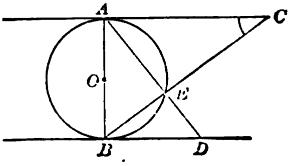
Suggestion. Prove that arc $AF =$ arc FC and arc $BD =$ arc DC. Compare the measures of $\angle\!\!\!s\ EBD$ and DEB.



3. Let CD be perpendicular to diameter AB at the center, and let FE be any chord. Show that $\angle F = \angle G$.

Suggestion. Compare arc DB with arc AD.

4. If tangents be drawn to a circle at the vertices of an equilateral triangle inscribed in it, prove that the triangle formed by the tangents is also equilateral.



5. The points of contact of two parallel tangents are connected with any point E of the circle as indicated. Prove that $\angle C = \angle BAE$.

PROPOSITION XIV. PROBLEM

241. *To find a point at given distances from two given points.* In other words: *To draw a triangle, given the three sides.*

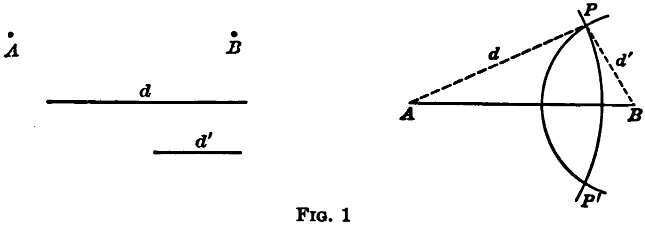


FIG. 1

Given two points A and B, and the distances d and d'.

Required to find a point at distance d from A, and at distance d' from B.

Construction. With A as a center and d as a radius, describe a circle. Then, with B as a center and d' as a radius, describe a circle.

Proof. If these arcs intersect, the points of intersection will be at distance d from A and distance d' from B. Why?

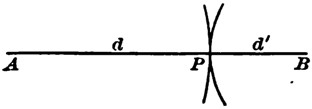


FIG. 2

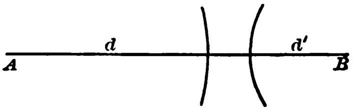


FIG. 3

Discussion. Under what conditions will there be:

1. Two points, P and P', as in Fig. 1?
2. One point P, as in Fig. 2?
3. No point P, as in Fig. 3?

The discussion shows that an actual triangle having the three given sides can exist only when they are of such lengths that the sum of any two is greater than the third.

242. COROLLARY. *To find a point that is at the same distance from each of two given points.*

This corollary may be stated: *To construct an isosceles triangle on a given base and having the equal sides of given length.*

243. Preparatory.

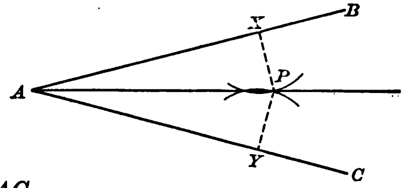
1. How many points are required to determine a straight line?

2. The vertex of an angle is a point of the bisector of an angle. How many points in addition to the vertex are necessary to determine the bisector?

3. Draw an acute angle and locate a point on each side two inches from the vertex. Find a point 1.5 in. from each of these two points.

PROPOSITION XV. PROBLEM

244. *To draw the bisector of a given angle.*



Given $\angle BAC$.

Required to bisect it.

Construction.

1. On the sides of $\angle BAC$ take points X and Y equally distant from A. How is this done?

2. With X and Y as centers and with any convenient radius, draw arcs intersecting at P.

3. Draw AP. This line bisects $\angle BAC$.

Proof. 1. $\triangle APX$ is congruent to $\triangle APY$. Sec. 107.

2. Therefore, $\angle CAP = \angle BAP$.

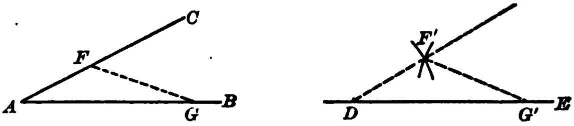
3. And $\angle BAC$ is bisected by AP. Sec. 37, Q.E.D.

PROPOSITION XVI. PROBLEM

245. *With a given line as one side, to construct an angle equal to a given angle.*

Given $\angle BAC$, and segment DE.

Required to construct an angle at D equal to $\angle BAC$.



Construction.

1. Take any two points, as F and G, one on each side of $\angle BAC$ and connect them by the line FG. This auxiliary line forms a triangle on which to base the construction.

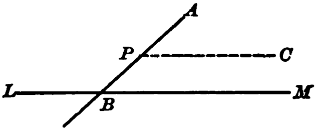
2. Lay off DG' on DE equal to AG. How?

3. On DG' construct $\triangle DG'F'$ congruent to $\triangle AGF$. Sec. 241

Proof. $\angle G'DF' = \angle GAF$ or $\angle BAC$. Sec. 94

Discussion. Another angle with vertex at D may be constructed on the lower side of DE.

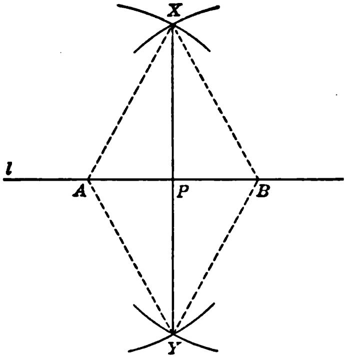
246. COR. 1. *Through a given point P to construct a parallel to a given line, LM.*



Construction. Through P draw an arbitrary line cutting LM at B. At P, construct $\angle CPA = \angle MBP$. Then PC will be the desired parallel. Prove it.

PROPOSITION XVII. PROBLEM

247. *To erect a perpendicular to a given straight line at a given point on it.*



Given the line l and the point P on l.

Required to draw a perpendicular to l at P.

Construction. 1. Take an arbitrary length PA and lay off PB equal to it. How?

2. With A as a center and AB as a radius describe an arc. With B as a center and BA as a radius describe an arc.

3. These arcs will intersect at points, as X and Y. Why?

4. Connect X with P and Y with P.

Proof. 1. $XP \perp AB$ and $YP \perp AB$ at P. Secs. 107. 101

2. $\therefore XY$ is a straight line $\perp AB$, or l, at P. Sec. 38. Q.E.D.

248. COROLLARY. *To bisect a given line segment.*

Observe that P bisects AB in the above figure. Make the construction.

EXERCISES

1. To divide a given angle into fourths.

2. At a given point on a line, to construct an angle equal to $1\frac{1}{2}$ right angles.

3. To construct a △ *ABC*, in which *AB* shall be 3 in., the angle at *A* equal to $\frac{1}{2}$ of a right angle, and the angle at *B* equal to $\frac{1}{4}$ of a right angle.

4. To construct an isosceles triangle with a given base and given angle at the base.

5. To construct an angle equal to twice a given angle.

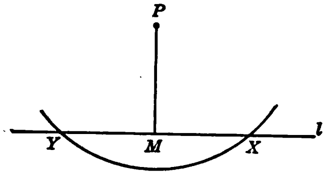
6. Given one of the acute angles of a right triangle and its adjacent leg, to construct the triangle.

7. Draw two parallels to a given straight line, one on either side of it.

8. Draw a parallel to a given line at a given distance from it.

PROPOSITION XVIII. PROBLEM

249. *To draw a perpendicular to a given straight line from a given point not on it.*

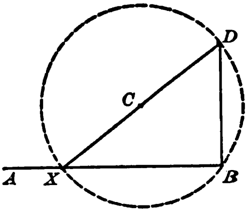


Given the line l and a point P not on l.

Required to draw a perpendicular from P to l.

Construction. 1. With P as a center and with a radius of sufficient length to cut l, describe a circle intersecting l at X and Y.

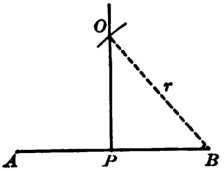
2. Draw the perpendicular bisector of XY. Secs. 247, 248.
Supply the proof.



250. Cor. 1. *To construct a perpendicular to a line segment at one of its end points.*

If the segment AB can be prolonged, apply Prop. XVII. If it cannot be prolonged, select a convenient point C, not in the segment, and with CB as a radius draw a circle. Let this circle cut AB at X. Draw the diameter XCD. Join B and D. Then $\angle DBX$ is inscribed in a semicircle, and is a right angle (Sec. 233). That is, DB is the desired perpendicular.

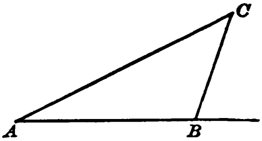
251. Cor. 2. *To draw a circle of given radius through two given points.*



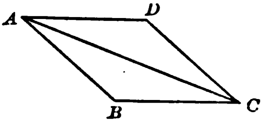
To do this, draw the perpendicular bisector of the line joining the two given points A and B. With one of the given points as the center and with the given radius as a radius, describe an arc cutting the perpendicular bisector, as at O. Why is O the center of the desired circle?

EXERCISES

1. What proposition shows how to construct the altitudes of a triangle?



2. In the figure, show how to construct the altitude of $\triangle ABC$ from C to AB. (It will meet AB at some point on its extension.)



3. In the figure, show how to construct the altitude of $\triangle ABC$ from A; also the altitude of $\triangle ADC$ from C.

4. Show how to construct a perpendicular at the end point of a given segment by erecting a perpendicular at some other convenient point of the segment and then using Sec. 246.

5. Show how to inscribe a circle in a given square. Also how to circumscribe a circle about the square.

6. Draw a triangle whose sides are 2, 2½ and 3 in. long.

7. Draw a triangle whose sides are 3, 4 and 5 in. long.

8. Mark two points A and B less than 4 in. apart. By drawing, find a point C, 2¼ in. from A and 2¼ in. from B.

9. Draw an equilateral triangle with sides 2 in. long.

10. Draw an isosceles triangle with base 3 in. and each of the other sides 1¾ in. long.

11. Draw an equilateral triangle with sides 1½ in. long. On each side of this triangle as base, draw an isosceles triangle, with the other sides 2 in. long.

12. Mark two points A and B less than 5 in. apart. Construct a point C that is 3 in. from A and 2½ in. from B.

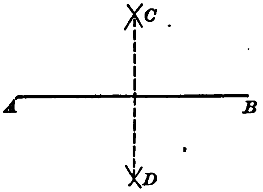
13. Mark two points A and B. Find a point C equally distant from A and B. Determine another point D equally distant from A and B. What relation has the line produced through C and D to the line AB?

14. Draw a straight line segment. On this as base construct an isosceles triangle with sides twice as long as the base.

15. Draw an equilateral triangle, with sides three times as long as a given line AB.

16. On a given base, draw an isosceles triangle with a given altitude.

17. Draw a straight line segment, and by compasses onl find points in its prolongation.

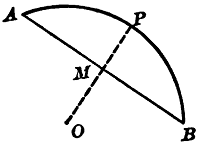


Suggestion. 1. Let AB be the given segment, and find C, D, points such that $AC = CB = BD = DA$. How are they found?

2. Then, lines AB and CD bisect each other at right angles. Therefore, any point equidistant from C and D is a point of AB, or of AB extended.

PROPOSITION XIX. PROBLEM

252. *To bisect a given arc.*



Given the arc AB.

To find its mid-point.

Construction. 1. Draw the chord AB and apply Sec. 248

2. Construct the perpendicular OM and apply Sec. 210

EXERCISES

1. In the figure $\triangle ABC$ is equilateral, and altitudes are drawn from A and B as indicated. The base AB is divided into four equal segments, and MO and NP are perpendicular to AB. O and P are the centers of the arcs BC and AC. Construct a similar arch on a base 2 in. long.

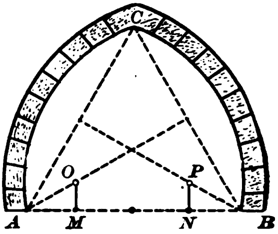


FIG. 1

2. Explain from the following figures how to construct the arches.

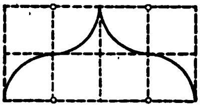


FIG. 2

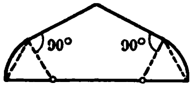


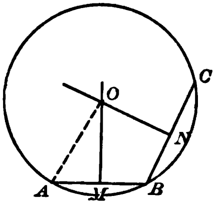
FIG. 3

3. Construct arches like Figs. 2, 3.

Arch 3 is called a *two centered arch.* What may arch 2 be called?

PROPOSITION XX. PROBLEM

253. *To construct the circle through three given points.*



Given three points A, B, C.

To construct the circle passing through A, B and C.

Construction. 1. Draw AB and BC.

2. Construct the perpendicular bisector of AB. Construct the perpendicular bisector of BC.

3. If A, B, C are not collinear, these perpendiculars will intersect as at O. Sec. 149

4. With O as the center and OA as the radius, describe a circle. It passes through A, B and C.

Proof. Apply Sec. 188.

If A, B and C are collinear, why is it impossible to draw a circle passing through them?

254. COR. 1. *To find the center for any given arc.*

Take any three points on the arc and construct the center of the circle through these points. Sec. 188

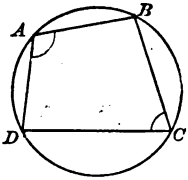
255. COR. 2. *To construct a circle through the three vertices of a triangle.*

Note. A circle can be circumscribed about every triangle but not about every quadrilateral. Why?

256. A quadrilateral about which a circle may be circumscribed is called an **inscriptible quadrilateral.**

PROPOSITION XXI. THEOREM

257. *If a quadrilateral is inscriptible, the sum of each pair of opposite angles is two right angles, and conversely.*



Given an inscriptible quadrilateral $ABCD$.

To prove that $\angle A + \angle C = 2$ right angles.

Proof. 1. $\angle C$ is measured by $\frac{1}{2}$ arc DAB. Why?

2. $\angle A$ is measured by $\frac{1}{2}$ arc BCD.

3. $\therefore \angle A + \angle C = 2$ right angles. Q.E.D. Why?

Show that $\angle B + \angle D = 2$ right angles.

Conversely, if the opposite angles of a quadrilateral are supplementary, the quadrilateral is inscriptible.

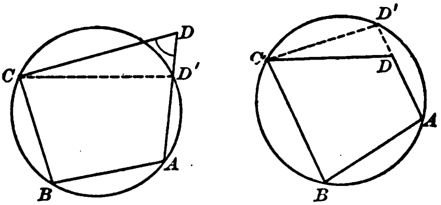


FIG. 1 FIG. 2

For, if it is not inscriptible, a circle passing through three vertices, as A, B and C, will not pass through D. Then, either Fig. 1 or Fig. 2 results. Connecting C with the point D' where AD cuts the circle, we have $\angle D' > \angle D$ in Fig. 1, and $< \angle D$ in Fig. 2, but supplementary to $\angle B$ in each case by the theorem. Therefore, the supposition that quadrilateral $ABCD$ is not inscriptible is incorrect, and the proposition is proved.

EXERCISES

1. How many angles of a quadrilateral can be chosen arbitrarily if it is to be inscriptible?

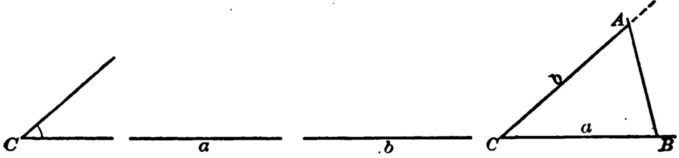
2. Choose as many angles of $ABCD$ arbitrarily (in degrees) as you can, and then determine the others so that the quadrilateral shall be inscriptible.

3. Given that quadrilateral $ABCD$ is inscriptible and that A contains $2x$ degrees and C, $3x$ degrees, B, $3y$ degrees and D, $7y$ degrees. Determine the number of degrees in each angle.

4. Given that the number of degrees in the angles of an inscriptible quadrilateral $ABCD$ are $A = 3x - y$, $B = 5x - 2y$, $C = 2x + 4y$, $D = 5x - y$. Determine the number of degrees in each angle.

PROPOSITION XXII. PROBLEM

258. *To construct a triangle given two of its sides and the angle included between them.*



Given the sides a and b, and their included $\angle C$.

To construct the triangle ABC.

Construction. 1. Draw the side a, and at its end C construct an angle equal to C. Sec. 245

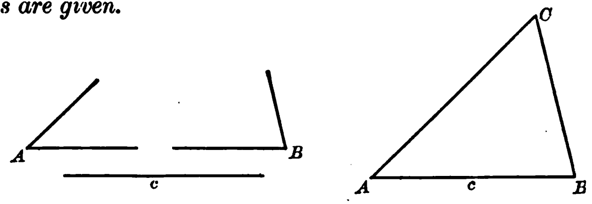
2. On the side of this angle lay off from C the side b. Call its end A.

3. Connect A with the end of side a, or point B.

4. $\triangle ABC$ is the required triangle.

Proof. $\triangle ABC$ has the sides a and b which include the given angle C. Q.E.D.

259. Cor. 1. *To construct a triangle when a side and two angles are given.*



If the two given angles are A and B, adjacent to the given side c, construct these angles by Sec. 245.

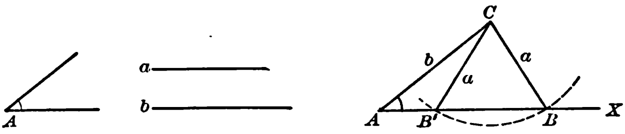
If one of the given angles is C opposite to the given side c, the third angle B can be found by constructing the difference between the sum of $\angle$s A and C and a straight angle.

260. Cor. 2. *To construct a parallelogram when two sides and their included angle are given.*

Apply Sec. 245, and draw lines parallel to the given sides.

Proposition XXIII. Problem

261. *To construct a triangle when two sides and an angle opposite to one of them are given.*



Given the sides a and b, and the $\angle A$ opposite to a.

To construct the triangle ABC.

Construction. 1. At A, the end of b, construct an angle equal to the given angle A.

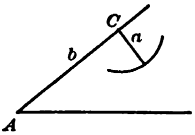
2. With the other end C of b as a center and with a radius equal to a, describe an arc cutting AX in B.

Connect C with B. Then, $\triangle ABC$ is the required triangle.

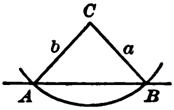
Discussion. 1. If a is less than b, as in the figure, the circle will cut AX again as at B', and there are two triangles satisfying the given conditions. This is called the **ambiguous case.**

2. If a equals the length of the perpendicular from C to AX, then the required triangle is a right triangle.

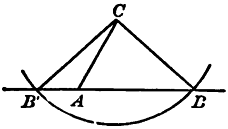
3. If a should be given less than the perpendicular distance from C to AX, the construction would be impossible. Similarly, if $a \leqq b$ and $\angle A$ is a right angle or an obtuse angle, the construction is impossible, for the side of a triangle opposite the right or obtuse angle is the longest side.



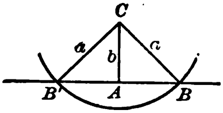
4. If $a = b$ and $\angle A$ is an acute angle, the arc cuts AX in A and B and there results one solution, an isosceles triangle.



5. If $a > b$ and $\angle A$ is an acute angle, the arc cuts AX in B' at the left of A. There is only one solution, ACB, because ACB' has an obtuse angle at A. Similarly, if $a > b$ and $\angle A$ is an obtuse angle, there is only one solution, that is, $\triangle AB'C$.



6. If $a > b$ and $\angle A$ is a right angle, two congruent right triangles satisfy the conditions.



262. Corollary. *To construct a right triangle, given the hypotenuse and one side.*

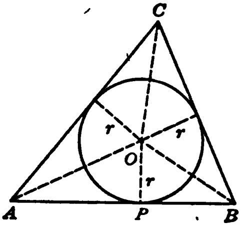
EXERCISES

1. If a diagonal of a quadrilateral bisects the two angles whose vertices it connects, that diagonal divides the quadrilateral into two equal triangles.

2. If from a point on the bisector of an angle, lines are drawn parallel to its sides, a rhombus is formed.

PROPOSITION XXIV. PROBLEM

263. *To inscribe a circle in a given triangle.*



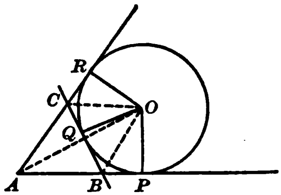
Given the triangle ABC.

To construct a circle tangent to AB, BC and AC.

Construction. 1. Draw the bisector of $\angle A$; also of $\angle B$. They will meet at point O. Sec. 177, 1

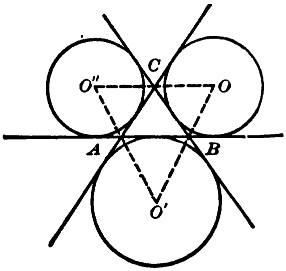
2. With O as the center and OP, the perpendicular from O to AB, as the radius, describe a circle. It will be tangent to AB, BC and AC.

Proof. Apply Sec. 177, 1 and Sec. 219.



264. COROLLARY. *To construct a circle tangent to one side of a triangle and to the other sides extended.*

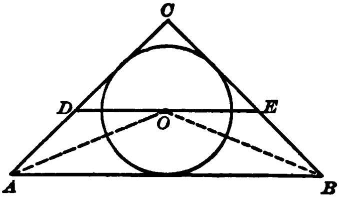
Draw the bisectors of two exterior angles, as B and C. Prove that they intersect. Then show that their intersection, O, is the center of a circle tangent to the three sides.



265. Three such circles can be drawn and they are called the *escribed* circles of the triangle, and their centers are called the *excenters* of the triangle.

EXERCISE

If through the center O of the circle inscribed in the triangle ABC a parallel to side AB be drawn, cutting the other sides at D and E, prove that $OD = AD$ and $OE = EB$, and hence $DE = AD + EB$.



Suggestion. How does AO divide angle A? Compare $\angle DOA$ with $\angle OAB$. What kind of a triangle is AOD?

PROPOSITION XXV. PROBLEM

266. *Through a given point to draw a tangent to a given circle.*

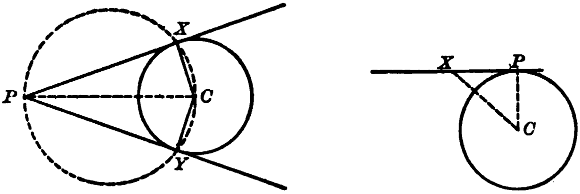


FIG. 1 FIG. 2

Given the point P, and the circle with center C.

To construct a straight line through P and tangent to $\odot C$.

CASE I. *P is not on the circle.*

Construction. 1. In Fig. 1, connect P with C.

2. Draw a circle with PC as diameter.

3. Connect P with X and Y, the intersections of the second circle with the given circle.

Proof. 1. PXC is a semicircle. $\therefore \angle PXC$ is a $\underline{|R}$. Why?

2. $\therefore PX$ and also PY is tangent to the circle. Sec. 219

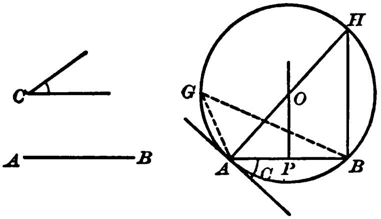
CASE II. *P is on the circle.*

In Fig. II, draw the radius CP, and $PX \perp CP$ at P. How?

Then, PX is tangent to the circle. Why?

PROPOSITION XXVI. PROBLEM

267. *On a given straight line as a base, to construct an arc in which a given angle can be inscribed.*



Given the straight line AB, and the angle C.

To construct an arc AHB such that $\angle H = \angle C$.

Construction. 1. Since AB is to be a chord of the desired arc, the center of the circle must lie somewhere on PO, the perpendicular bisector of AB. Sec. 187

Construct the perpendicular bisector PO.

2. Further, if we had the desired arc, as $AGHB$, the angle between the tangent at A and the chord AB would be equal to the given angle. $\therefore$ construct $\angle C$ at point A.

3. The center must lie somewhere on AH the perpendicular to the tangent at A. $\therefore$ construct $AH \perp$ the tangent. The center is, therefore, at O, the common point of AH and PO.

4. With O as the center and OA as a radius, describe the arc on AB.

268. COROLLARY. *An angle inscribed in the minor arc AB is supplementary to $\angle C$.*

269. An arc in which a given angle can be inscribed is said to be an arc *containing* the given angle.

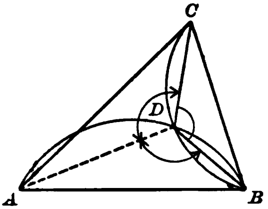
EXERCISES

1. With ruler and compasses draw an angle of 60°; upon a given line segment draw an arc in which the angle can be inscribed.

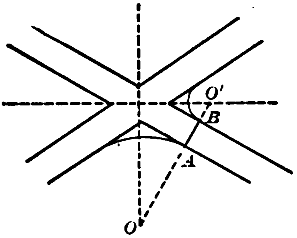
2. Draw an arbitrary quadrilateral *ABCD*. Upon *BC* erect an arc containing the angle *D*. How will the accuracy of your work be checked?

3. The apparent size of an object is measured by the angle between the rays of light from its extremities to the eye of the observer. Where must an observer be within the plane of a triangular field in order that the three sides may seem equally long to him?

Suggestion. Each side must subtend at the eye of the observer an angle of 120°. Construct an arc on each of two sides of the $\triangle$ in which an angle of 120° can be inscribed. Since for any point D within the triangle $\angle CDB > \angle CAB$, it follows that the problem admits no solution in case $\angle CAB \geqq 120°$. Similarly for $\angle s$ B and C.



4. In planning a park, it is desired to round off the corners of two intersecting roads with circular arcs tangent to the sides. The curves are to begin at *A* and at *B*, opposite ends of a perpendicular to the sides of the road. Show how to lay out the curves.

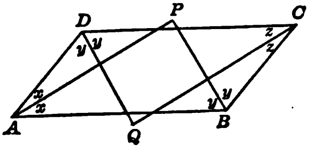


5. The bisectors of the angles of a parallelogram produced until they meet form a rectangle.

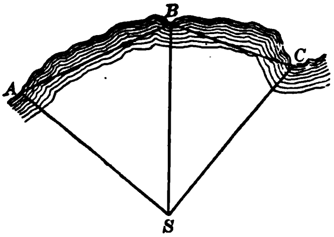
Suggestion. 1. $\angle 2x + \angle 2y = 180°$. Why?

2. $\therefore \angle x + \angle y = 90°$.

3. $\therefore$ in $\triangle ABP$, $\angle P = 90°$.



6. A sailor in a ship at S knew from his charts that the distance between two lighthouses, A and B, was five miles, and that between B and C, three miles. He also knew the angle ABC to be 160°. By measurement with his instruments he found $\angle ASB = 70°$, $\angle BSC = 35°$. By making a drawing of A, B, C to scale, and constructing on AB an arc in which the angle 70° could be inscribed, and on BC an arc in which 35° could be inscribed, he located S and measured by scale his approximate distance from B. Make the construction by which he located S.



270. Methods of solving Problems of Construction. No infallible directions can be given for solving problems that call for construction with ruler and compasses, but the following methods of attack are useful:

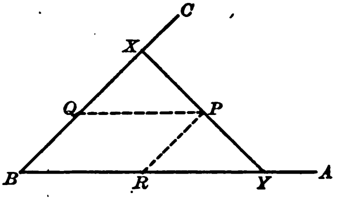
1. **The Synthetic.** This method depends upon the direct application of a known property, and does not cover a broad field.

Example. *To construct a tangent to a circle at a point on the curve.*

The known property here is that a tangent to a circle is perpendicular to the radius drawn to the point of contact. Hence, to construct the tangent, draw a radius to the given point P and then draw a perpendicular to the radius at P.

2. **The Analytic.** This method is based upon supposing the construction made and then investigating the figure to see what properties exist that would lead to the construction.

Example. *To draw a line through a given point within an angle so that the point will bisect the segment of the line terminated by the sides of the angle.*



In the figure, suppose the line drawn through the given point P so that $PX = PY$. Then, by drawing the auxiliary lines PQ and PR parallel to the sides of the angle, it appears that $\triangle YPR$ and $\triangle PXQ$ are congruent; hence, $BQ = QX$. The discovery of this property suggests at once the construction:

(1) Through P draw a line $PQ \parallel AB$.
(2) Lay off $QX = BQ$.
(3) Draw XP and extend to Y.

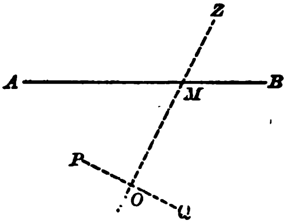
3. **The Intersection of Loci.** This method, as its name suggests, applies to problems where a point is required that satisfies two or more conditions, each depending upon some locus. Such problems constitute a special class.

Examples. 1. *To find a point in a given line that shall be equidistant from two given points.*

(1) The given line AB is one locus where the required point must be.

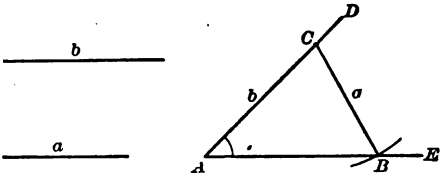
(2) The perpendicular bisector OZ of the line joining the given points P and Q is the other locus of the required point.

(3) Hence, the intersection of OZ and AB, namely M, is the required point.

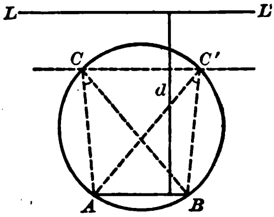


2. *To construct a triangle, given a, b, A.*

Let DAE be the given angle A. On the side AD, lay off AC equal to the given side b. Then AE is a first locus for the third vertex B. With C as a center and a as a radius, describe an arc cutting AE. This is the second locus. Their intersection is the location of the third vertex. How many such vertices may there be?



271. Number of Solutions. In the discussion of problems previously treated, we have noted that problems sometimes have more than one solution, also that the data may be such that no construction is possible. If there is a *definite* number of solutions, the problem is said to be *determinate*. If there is an *indefinite* number of solutions, the problem is said to be *indeterminate*. If there is *no solution*, the problem is said to be *impossible*.



Example. *To find the points at a given distance from a segment AB, such that the angle subtended by AB at these points equals a given angle, C.*

1. This problem is determinate if a parallel to AB at distance d from it is tangent to the arc on AB that contains the angle C, (Sec. 269) or cuts it, at CC'. If AB is not a diameter, the number of solutions is 1 and 2 respectively. If AB is a diameter, there are 2 and 4 solutions respectively.

2. This problem is impossible if the parallel named does not meet the arc, but is located like LL'.

3. If no distance d is given, the problem is indeterminate, for any point of the arc may serve as P.

272. Notations Used in Triangles. In a triangle ABC, the median to the side a is often denoted by m_a, that to side b is denoted m_b, and that on the side c by m_c. The altitude on the side a is denoted by h_a, and so on.

EXERCISES

Make the following constructions, using the method that seems best:

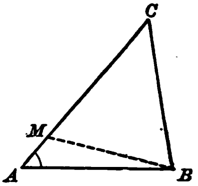
1. From a given point outside of a circle, draw a line to the nearest point on the circle; also draw a line to the farthest point on the circle.

2. Draw a line tangent to a given circle and parallel to a given line.

3. Construct a square having a given line segment as diagonal.

4. Construct a triangle, given one side, an angle adjacent to it, and the difference of the other two sides

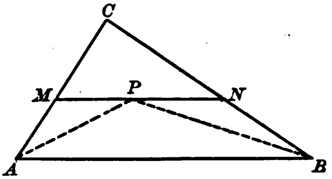
Suggestion. Given side AB, $\angle A$, and AM, the difference between AC and BC.



5. In triangle ABC draw a line $MN \parallel AB$ and equal to $AM + NB$.

Suggestion. What kind of triangles are APM and BPN, if $MP = AM$, and $NB = PN$? Compare $\angle MAP$ and $\angle PAB$.

6. Knowing the side a and m_a of a triangle ABC, what is a locus for the vertex A?



7. Construct a triangle, given a, B, and m_a.

8. Construct a triangle, given a, b, and m_a.

9. Construct a triangle on a given base with a given altitude, and having its third vertex lying on a given line.

10. Construct a triangle, given a, h_a, m_a.

11. Construct a triangle, given a, B, h_a.

12. Construct a triangle, given a, b, h_a.

13. Find the locus of the mid-point of a chord of given length, taking all possible positions in a given circle.

14. Find the locus of the mid-point of a chord passing through a fixed point, and taking all possible positions in a circle.

15. Find a point at a given distance from two given points.

16. Find a point equidistant from two intersecting lines and at a given distance from a given point.

17. Show how to construct a circle of given radius and tangent to:

(1) Each of two intersecting straight lines.

(2) A given straight line and a given circle.

(3) A given circle.

18. Show how to construct a circle of given radius passing through a given point and tangent to:

(1) A given straight line.

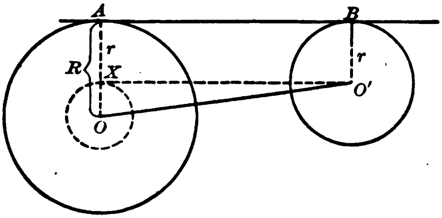
(2) A given circle.

19. Is it possible to construct a circle tangent to each of three given lines of which two are parallel and cut by the third? To three parallels? To three lines intersecting in the same point? In each case either show how to make the construction or show that no such circle exists.

20. Show how to construct a line segment of given length, extending in a given direction and having its end points on two intersecting straight lines.

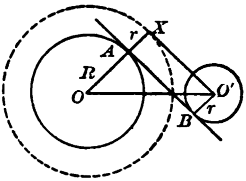
21. Construct a common tangent to two given circles.

Suggestion. Assume the tangent drawn and observe that $AOO'B$ is a trapezoid in which A and B are right angles. We could construct this figure if we knew $\angle O$. Draw the auxiliary line $O'X \perp R$, and observe that in $\triangle OO'X$, we know OO' and XO, which is $R - r$, and $\angle X =$ a right angle. Therefore, construct $\triangle OO'X$ and extend OX to A. Draw $AB \parallel XO'$.



22. In the figure OX is the sum of the radii R and r of the circles with centers O and O'. Show that AB is a common tangent to these circles and show how to construct it.

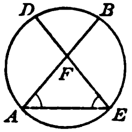
23. Show that if two circles lie completely without each other with no point in common, they have four common tangents.



24. How many common tangents have two circles that are tangent externally? Internally? That intersect in two points? Such that one lies within the other with no common point? Make a figure and give the proof for each case.

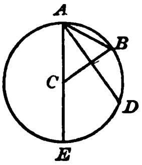
REVIEW EXERCISES

1. If two equal chords, AB and DE, intersect in the point F, show that $AF = FE$.

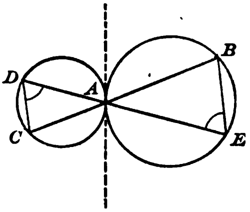


Suggestion. Show that $\angle A = \angle E$ by referring to equal arcs.

2. If arc AB contains 36° and AD bisects $\angle CAB$, prove that $\angle CAD = 36°$.



3. If two secants be drawn through the point of contact of two tangent circles, the chords joining the other extremities of the secants are parallel.

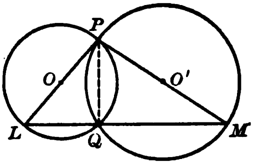


Suggestion. To prove $DC \parallel BE$, it is sufficient to prove that $\angle D = \angle E$. Why?

$\angle D$ is measured by $\frac{1}{2}$ arc AC and $\angle E$ is measured by $\frac{1}{2}$ arc AB. But these arcs are equal, since they subtend equal vertical angles at A.

4. If at points of the base of an isosceles triangle equidistant from its mid-point, perpendiculars be erected, terminating in the sides of the triangle, and if the end points of the perpendiculars be joined, the figure so formed is a rectangle.

5. If from one point of intersection of two intersecting circles the diameters be drawn, prove that the other intersection lies on the straight line connecting the other extremities of the diameters.

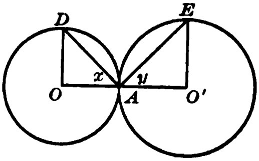


Suggestion. Connect Q with L and with M. Prove $\angle LQM$ to be a straight angle.

6. Parallel chords drawn from the extremities of a diameter are equal.

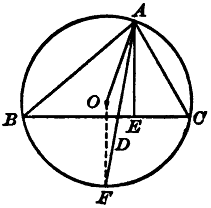
Prove that the line joining the other extremities of the chords is a diameter.

7. If two circles are tangent externally at the point A, and if two chords meeting at A at an angle of 90° cut the circles again in D and E, prove that the radii to D and E are parallel.



Suggestion. $\angle$s x and y are complements. $\angle O$ is a supplement of $2x$. $\angle O'$ is a supplement of $2y$. Show that $\angle$s O and O' are supplements.

8. Prove that in any triangle inscribed in a circle, the bisector of the angle at any vertex bisects also the angle between the altitude from that vertex and the radius to that vertex.



Suggestion. If DA bisects $\angle BAC$, to show that it bisects $\angle OAE$. Since AD bisects $\angle BAC$, arc $BF =$ arc FC. $\therefore OF$ is $\perp BC$. $OF \parallel AE$. (Why?) And $\triangle FOA$ is isosceles.

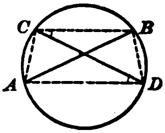
9. Draw a figure and apply the proof of Exercise 8 to the case where C is an obtuse angle.

10. Given the perimeter of a rectangle 90 in., and its altitude $\frac{7}{8}$ of its base. Find its dimensions.

Suggestion. $2(x + \frac{7}{8}x) = 90$.

11. If two equal chords intersect, their end points are the vertices of an isosceles trapezoid inscribed in the circle.

Suggestion. Arc $CD =$ arc AB. Why?
Arc $CA =$ arc DB. Why?
Chord $CA =$ chord BD. Why?
$\angle C = \angle D$. Why?
$AD \parallel CB$. Why?



12. Show that the common tangent at the point of contact of two tangent circles is perpendicular to the line of centers.

13. Show that two circles that are tangent to the same straight line at the same point are tangent to each other, and conversely.

Suggestion. Suppose the circles to have a second point in common. Draw their common chord. What relation must this line of centers have to this chord? Can it have this relation according to the hypothesis?

14. Show that a circle drawn on AB as a diameter in any one of the four figures will be tangent to the two given circles having C and O as centers.

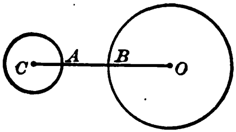


FIG. 1

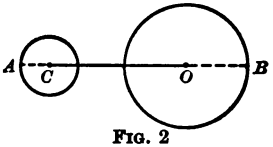


FIG. 2

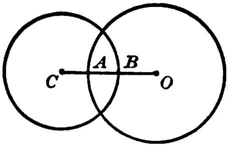


FIG. 3

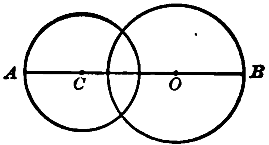


FIG. 4

Suggestion. Draw the common tangent at A and B and apply Exercise 13.

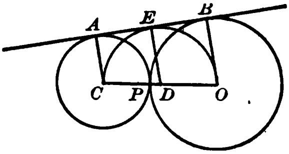
15. If two circles are tangent externally, prove that the sum of the bases of the trapezoid formed by a common exterior tangent, the radii to the points of contact and the line of centers, is equal to the distance between the centers.

16. If two circles are tangent externally, prove that the circle described on the line of centers as diameter is tangent to the common exterior tangents of the two circles.

Suggestion. From D, the mid-point of CO, draw $ED \parallel OB$. Then

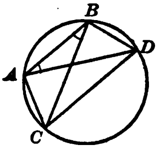
$$\begin{aligned} DE &= \tfrac{1}{2}(AC + BO) \\ &= \tfrac{1}{2}(CP + PO) \\ &= DO. \end{aligned}$$

$\therefore$ DE is a radius. It is $\perp$ AB by construction.



17. From the extremities of a chord and on the same side of it, two chords are drawn making equal angles with the given chord; prove (1) that the two chords are equal; (2) that the chord joining their other extremities is parallel to the first chord.

Suggestion. Compare arcs AC and BD. Then, arcs BAC and ABD are equal.



18. What is the locus of the mid-points of all chords which, when extended, pass through a given point without the circle?

19. If the pairs of opposite sides of an inscribed quadrilateral are extended until they intersect, the bisectors of the angles thus formed intersect at right angles.

Suggestion. (The lines indicated are arcs throughout.)

(1) $NB - QC = AN - DQ$. Why?

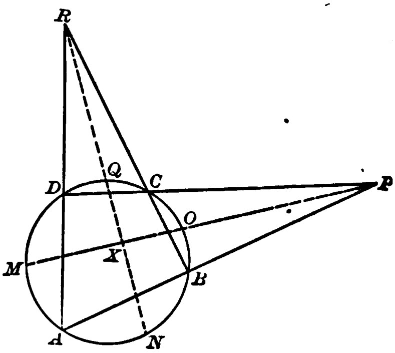
(2) $AM - BO = DM - CO$. Why?

(3) $AN + QC = NB + DQ$. From (1)

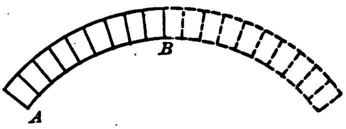
(4) $AM + CO = DM + BO$. From (2)

(5) $NM + OQ = ON + QM$. Adding (3) and (4)

$\therefore$ $\angle OXN = \angle NXM$. Why?

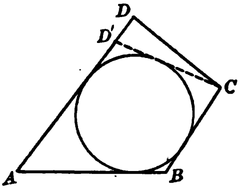


20. An engineer is asked to extend the circular arc *AB* of the railway curve indicated in the figure by a length equal to *AB*. How can he find the center for the arc *AB* and stake off the extension?



21. The bisectors of the angles of a circumscribed quadrilateral pass through a common point.

22. If the sum of a pair of opposite sides of a quadrilateral equals the sum of the other pair, the quadrilateral is circumscriptible.



Suggestion. Suppose that the circle tangent to the other three sides is not tangent to *CD*. Then it is to some line, as *CD'*.

Then, $D'C + AB = BC + AD'$. Ex. 7, p. 129

And $DC + AB = BC + AD$. Given

$\therefore DC - D'C = AD - AD' = D'D$.

But this is impossible by Sec. 80, IV.

Assume that the circle cuts the line *CD* and draw a tangent *CD'*. Prove this case similarly.

23. If two tangents intersect at an angle of 50°, how many degrees in their minor intercepted arc?

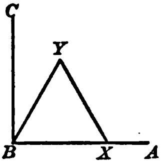
24. If two equal circles are tangent externally, their common internal tangent bisects their common external tangents.

25. Construct a triangle, given its base, the vertical angle and the median to the base. Discuss the possible cases.

26. Two circles whose radii are 20 in. and 12 in. respectively are tangent externally. Find the length of their line of centers. What would be the length if the circles were tangent internally?

27. If a common tangent to two unequal circles cuts the line of centers at the point *P*, a tangent to one of the circles passing through *P* will be tangent to the other.

28. Trisect a right angle.



Suggestion. Choose an arbitrary length BX on one side and construct an equilateral triangle BXY. Bisect $\angle XBY$.

29. Trisect a circle.

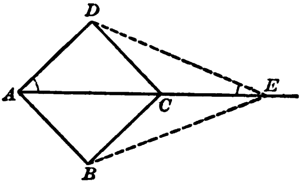
30. Construct a square having given its perimeter.

31. Construct a square having given its diagonal.

32. Construct an angle of $22\frac{1}{2}°$.

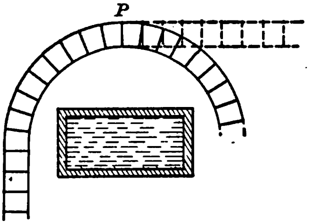
Suggestion. Bisect a right angle. Bisect one of these parts.

33. Construct a square having given the sum of a diagonal and one side.

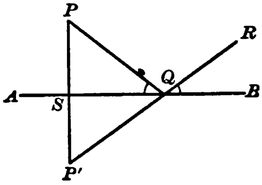


Suggestion. Given AE, the sum of the diagonal and a side. Construct at A an angle of 45° and at E an angle of $22\frac{1}{2}°$.

34. It is desired to change the railroad curve at P into a straight track tangent to the rest of the curve. The center of the curve is inaccessible on account of a reservoir. Show how to construct the tangent.



35. Draw two lines from two given points P and R which shall meet a given line AB at one point, as Q, and make equal angles with AB.



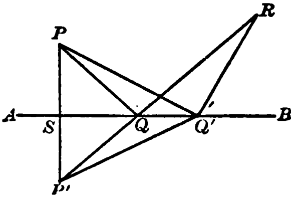
Suggestion. Draw $PS \perp AB$, and extend it to P' so that $SP' = SP$. Connect P' with R. Q is the required point. Why?

36. Find the shortest path from a point P to a line AB and thence to a point R on the same side of AB.

Suggestion. In the figure $PS \perp AB$ and $PS = SP'$.

$$PQ + QR = P'Q + QR.$$
$$P'Q + QR < P'Q' + Q'R.$$

State the construction.



The path of a ray of light from a point to a plane mirror and thence to another point makes equal angles with the mirror; hence, by Exercise 36, it takes the shortest possible path.

EXERCISES

1. If a median of a triangle is half the side to which it is drawn, the triangle is a right triangle.

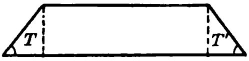
2. The perpendicular bisectors of the two legs of a right triangle intersect on the hypotenuse.

3. If one leg of a right triangle is half the hypotenuse, the included angle between them is 60°, and conversely.

4. Show that, given any point C within a right angle, a line can always be drawn through C such that C is the mid-point of the segment of the line comprised between the sides of the right angle.

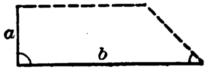
5. If the two angles at the end of one base of a trapezoid are equal, the trapezoid is isosceles.

Suggestion. Drop perpendiculars; prove $\triangle$s T and T' congruent.



6. In an isosceles trapezoid, the two angles at the base are equal.

7. Construct a trapezoid, given the altitude, the longer base, and the angles adjacent to this base, of which one is a right angle.



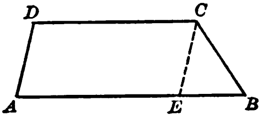
8. Construct a trapezoid, knowing the four sides.

Suggestion. Assume the figure constructed and draw $CE \parallel DA$. Then,

In $\triangle CBE$, three sides are known, $\therefore$ we can construct it.

Then prolong BE to A, such that BA is the given base.

Through C draw a parallel to AB, and lay off CD equal to the other given base.

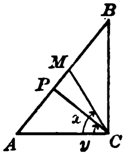


9. In a right triangle, the median and the altitude drawn from the vertex of the right angle, form an angle equal to the difference of the acute angles.

Suggestion. 1. $\angle x = \angle A$. By Ex. 8, p. 135.

2. $\angle y = \angle B$. Sec. 142.

3. Subtract (2) from (1).



10. Prove that a circle erected on one leg of an isosceles triangle as diameter, bisects the base.

11. Show how to construct an isosceles triangle, given the base and the radius of the inscribed circle.

12. Two tangents are drawn to a circle from a point without it; prove that the triangle formed by these tangents and any tangent to the arc included by them, has a perimeter equal to the sum of the first two tangents.

13. Show how to construct a circle passing through a given point, and tangent to a given circle at a given point on it.

14. Prove that if a tangent is drawn at an extremity of any chord of a circle, the perpendiculars to the tangent and the chord from the mid-point of the arc subtended by the latter, are equal.

15. Prove that if the inscribed and circumscribed circles of a triangle are concentric, the triangle is equilateral.

16. Prove that in any right triangle, the sum of the hypotenuse and the diameter of the inscribed circle is equal to the sum of the legs of the triangle.

17. Show how to construct a trapezoid when the four sides are given.

18. Two equal chords produced meet outside the circle; prove that the secants thus formed are equal.

HISTORICAL NOTE

DESCARTES

The idea of a locus was known to the ancients. Proclus, in the fifth century A.D., who wrote a commentary on Euclid, and to whom we owe much of our knowledge of that work, spoke of loci saying, "I call locus theorems those theorems in which the same property is found to exist throughout the whole of some locus."

RENÉ DESCARTES

The conditions to be satisfied by the points of a locus are, of course, geometric conditions, and for many centuries they were expressed in geometric terms only. It remained for Descartes, in the seventeenth century, to discover a method of expressing the conditions in the form of an algebraic equation which is known as the equation of the locus. This method is the basis of graphic algebra, where we learn to know equations of the simplest loci, the straight line and the circle. The systematic study of the properties of geometric figures by means of their equations is known as Analytic Geometry. Here the methods of algebra are applied to the science of geometry in a marvelously effective way and the creation of this branch of mathematics brought Descartes lasting fame as a mathematician.

René Descartes was born at La Haye in 1596 A.D. At the age of twenty-one he enlisted as a soldier under Prince Maurice of Orange, pursuing the study of mathematics in his leisure hours. In 1629 he went to Holland, Dutch culture and learning then being at its height, and there devoted twenty years to the preparation of his works in mathematics and philosophy. Among the many devotees and admirers of Descartes was Queen Christina of Sweden, who finally prevailed upon him to visit the Swedish Court. After a short stay, he died at Stockholm in 1650.

BOOK III

MEASURES, PROPORTION AND SIMILAR FIGURES

273. Preparatory.

A 20-inch segment contains a 1-inch segment 20 times. The 1-inch segment is called a measure of the 20-inch segment. Likewise, a 2-inch segment is a measure of a 20-inch segment, and is contained in it a whole number of times, namely, 10 times. A $\frac{1}{4}$-inch segment is also a measure of a 20-inch segment, for it is contained in the given segment 80 times.

But a 3-inch segment or a $\frac{6}{7}$-inch segment is not a measure of a 20-inch segment, since neither is contained in it an integral number of times.

274. Meaning of Measure. If a segment, S, is the sum of an integral number, n, of equal segments, u, the segment u is said to be contained n times in S. The segment u is also said to be a **measure** of S.

EXERCISES

1. Name three segments that are measures of a 12-inch segment and three that are not.

2. Name five segments that are measures of a 1-yard segment and five that are not, including in each set at least one whose length is a fractional number of inches.

3. Name three segments that are measures of a $22\frac{1}{2}$-inch segment, and three that are not.

275. Preparatory. A 4-inch segment is called a common measure of a 24-inch and a 36-inch segment, because it is contained a whole number of times in each, namely 6 times in the first, and 9 times in the second. A 1-foot segment is also a common measure, being contained 2 times in the first segment and 3 times in the second. A $\frac{1}{5}$-inch segment and a 12-inch segment are also common measures.

But an 8-inch segment is not a common measure, nor is a $\frac{3}{4}$-foot segment.

276. Common Measure. A segment that is contained an integral number of times in each of two segments *a* and *b*, is called a **common measure** of these segments.

EXERCISES

For each of the following pairs of segments, name three segments that are common measures and three that are not, the latter being, in each instance, a measure of one of the segments of the pair:

1. 8 ft. 6 ft.
2. 5 in. 7 in.
3. 7.2 mi. 3.9 mi.
4. 1 mi. 1 in.
5. 7000 ft. .0026 ft.
6. 1.37 in. .29 in.

277. Preparatory. Segments exist that have no common unit of measure. For example, it can be proved that in an isosceles right triangle the hypotenuse and a side have no common measure.

278. Commensurable Magnitudes. Two segments that have a common measure are called **commensurable.**

279. Incommensurable Magnitudes. Two segments that have no common measure are called **incommensurable.**

Secs. 278, 279 will apply as well to arcs and angles as to segments. All that need be done is to replace the word "segments" wherever it occurs by the word "arcs" or the word "angles" as the case may be.

280. Preparatory. In arithmetic and algebra the *ratio* of two like numbers (integers or fractions) has already been defined as their quotient.

The numbers must both be abstract, or both concrete of the same kind. The ratio itself is always an abstract number.

Thus, the ratio of 12 to 3 is 4; the ratio of 12 to 8 is $\frac{3}{2}$; the ratio of 9 to 16 is $\frac{9}{16}$; the ratio of $\frac{3}{4}$ to $\frac{5}{7}$ is $\frac{21}{20}$; the ratio of 12 qt. to 2 qt. is 6; the ratio of 22 in. to 5 in. is $\frac{22}{5}$.

281. Ratio. The ratio of two commensurable line segments is the *ratio of the numbers* which state how many times the given segments contain a common measure.

A similar definition applies to any other magnitudes for which common measures have been defined, for example, arcs or angles.

When we say in physics that weight (w) varies as density (d) and write $\frac{w}{d}=\frac{w'}{d'}$ we mean that the ratios of the abstract numbers measuring these quantities are equal.

282. Incommensurable Ratio. The ratio of two incommensurable magnitudes might also be defined and called an **incommensurable ratio.** The treatment of such ratios is, however, a difficult matter and beyond the province of a school course in elementary geometry. We shall therefore confine our proofs involving ratios to the case where the magnitudes (segments, arcs, angles, etc.) are commensurable. Further, it is always possible to find commensurable magnitudes that shall differ from the given magnitudes by as little as may be desired; that is, that shall be practically equal to them.

For example, suppose that we wish to find two commensurable line segments differing from the given segments by less than $\frac{1}{100}$ in. each. With $\frac{1}{100}$ in. as a unit, measure the first line. Either the measurement comes out exact, in which case $\frac{1}{100}$ in. is a measure of the line, and we use the given segment as one of the new segments, or there finally remains a small part of the line less than $\frac{1}{100}$ in. long. Cutting off this part, we have left a segment differing from the given one by less than $\frac{1}{100}$ in. and having $\frac{1}{100}$ in. as a measure.

In the same way we could find two segments each differing from one of the given segments by less than $\frac{1}{1,000,000}$ in. and having $\frac{1}{1,000,000}$ in. as a measure.

The error committed by taking commensurable magnitudes in place of incommensurable ones can thus be made as small as we please, that is, practically negligible, and consequently it is sufficient for all practical purposes to consider only commensurable magnitudes.

283. Notation for Ratios. Ratios are commonly indicated by the fractional notation.

Thus, the ratio of 2 to 3 is written $\frac{2}{3}$; the old form $2:3$ is less convenient in calculation.

When letters are used, they may denote numbers measuring magnitudes of any sort for which the term ratio has been defined.

284. Proportion. An equation between two ratios is called a **proportion.**

Thus, $\frac{2}{3}=\frac{6}{9}$, $\frac{3a}{7a}=\frac{6b}{14b}$, $\frac{25}{5}=\frac{5}{1}$, $\frac{5}{6}=\frac{3}{x}$, $\frac{a}{b}=\frac{c}{d}$ are proportions.

285. Means and Extremes. The older form of writing proportions is: $a:b::c:d$, from which a and d are called the extremes and b and c the means. When so written the proportion is read: "a is to b as c is to d."

The numbers forming one of the two equal ratios of a proportion are said to be proportional to the numbers forming the other.

Thus, 12 and 9 are proportional to 20 and 15 because $\frac{12}{9}=\frac{20}{15}$.

286. Fourth Proportional. In the proportion $\frac{a}{b}=\frac{c}{d}$, the fourth number, d, is called the **fourth proportional** to the other three.

287. Mean Proportional. In the proportion $\frac{a}{b}=\frac{b}{c}$, b is called the **mean proportional** between a and c.

The mean proportional between a and c equals the square root of their product. For $b^2 = ac$, hence, $b=\sqrt{ac}$.

288. Relation of Proportion to Equations. The properties of equations apply to a proportion.

PROPOSITION I. THEOREM

289. *If* a, b, c, d *are numbers, and if* $\frac{a}{b}=\frac{c}{d}$, *then:*

I. $ad = bc.$ II. $\frac{a}{c}=\frac{b}{d}.$ III. $\frac{ma}{b}=\frac{mc}{d}.$

IV. $\frac{a^n}{b^n}=\frac{c^n}{d^n}.$ V. $\frac{\sqrt[n]{a}}{\sqrt[n]{b}}=\frac{\sqrt[n]{c}}{\sqrt[n]{d}}.$ VI. $\frac{a+b}{c+d}=\frac{b}{d}.$

VII. $\frac{a-b}{c-d}=\frac{b}{d}.$ VIII. $\frac{a+b}{a-b}=\frac{c+d}{c-d}.$ IX. $\frac{b}{a}=\frac{d}{c}.$

Proof. For convenience let the given equation be denoted by *E*. To obtain the results above proceed as follows:

I. Multiply both members of *E* by *bd*.

II. Multiply both members of *E* by $\frac{b}{c}$.

III. Multiply both members of *E* by *m*.

IV. Raise both members of *E* to the *n*th power.

V. Take like *n*th roots of both members of *E*.

VI. Adding 1 to both members of *E* we have $\frac{a}{b}+1=\frac{c}{d}+1$ or $\frac{a+b}{b}=\frac{c+d}{d}$. Whence $\frac{a+b}{c+d}=\frac{b}{d}$.

VII. Subtract 1 from both members of *E* and follow VI.

VIII. Equate the two values of $\frac{b}{d}$ found in VI and VII and apply II.

IX. The reciprocals of equal ratios are equal.

NOTES. — 1. The above results are sometimes stated in words, thus:

I. The product of the means equals the product of the extremes. III, IV, V, the same multiples (or powers or roots) of equal ratios are equal.

2. Certain of the above results are sometimes said to be obtained from the original proportion as follows:

II, by *alternation;* VI, by *composition;* VII, by *division;* VIII, by *composition* and *division* and IX, by *inversion.*

EXERCISES

Find x, given that:

1. $\frac{3}{4}=\frac{x}{12}$. **3.** $\frac{x+1}{x+2}=\frac{5}{7}$. **5.** $\frac{4}{x+2}=\frac{x+4}{12}$.

2. $\frac{5}{x+1}=\frac{4}{3}$. **4.** $\frac{3}{x}=\frac{x}{12}$. **6.** $\frac{x+1}{x+2}=\frac{x+5}{x+3}$.

Given that $\frac{x}{y}=\frac{2}{3}$, show that:

7. $\frac{x^2}{y^2}=\frac{4}{9}$. **9.** $\frac{x+2}{y+3}=\frac{2}{3}$.

8. $\frac{y}{x}=\frac{3}{2}$. **10.** $\frac{x-2}{y-3}=\frac{x}{y}$.

11. If $y=4$, find x from (9) and (10), and compare their values.

12. Show that if the product of two numbers equals the product of two other numbers, either pair may be made the means and the other the extremes of a proportion.

Write each of the following as proportions in two different ways:

13. $3\times 12=4\times 9$. **14.** $5a=16$. **15.** $ap=qx$.

Proposition II. Theorem

290. If $\frac{a}{a'}=\frac{b}{b'}, \frac{a}{a'}=\frac{c}{c'}, \frac{a}{a'}=\frac{d}{d'}\cdots$, then $\frac{a}{a'}=\frac{a+b+c+d}{a'+b'+c'+d'}\cdots$;

or, *In a set of equal ratios the sum of the numerators divided by the sum of the denominators equals any one of the ratios.*

Proof. 1. Let $\frac{a}{a'}=r$, then $\frac{b}{b'}=r, \frac{c}{c'}=r$, etc.,

or, $a=a'r,\ b=b'r,\ c=c'r$, etc. 78, IV

2. Adding, $a+b+c\cdots=(a'+b'+c'+\cdots)r$, 78, II

or, $\frac{a+b+c+\cdots}{a'+b'+c'+\cdots}=r=\frac{a}{a'}$. 78, V, Step (1)

EXERCISES

1. From $\frac{2}{3} = \frac{4}{6}$, and $\frac{2}{3} = \frac{10}{15}$, show that $\frac{2}{3} = \frac{14}{21}$.

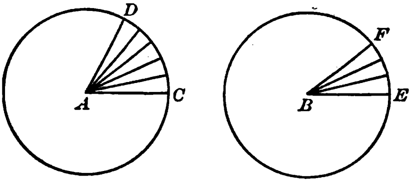
2. If $\frac{x}{5} = \frac{y}{7}$, $\frac{x}{5} = \frac{3y - 2}{20}$, and $\frac{x}{5} = \frac{4y + 4}{30}$, show that $\frac{x}{5} = \frac{8y + 2}{57}$.

3. If $\frac{a}{b} = \frac{b}{c}$, show that $\frac{a}{c} = \frac{a^2}{b^2}$.

Suggestion. 1. From the given equation, $ac = b^2$.
2. But $a^2 = a^2$.
3. Divide 2 by 1.

Proposition III. Theorem

291. *Two central angles are in the same ratio as their intercepted arcs.*



Given angles A and B at the center of equal circles, and CD and EF, their intercepted arcs.

To prove that $\frac{\angle A}{\angle B} = \frac{\text{arc } CD}{\text{arc } EF}$.

Proof. 1. A common unit of measure can be selected for angles A and B which will give their ratio within any desired degree of accuracy. Sec. 282

2. Let this unit angle be contained m times in A, and r times in B. Then $\frac{\angle A}{\angle B} = \frac{m}{r}$.

3. But, since the unit angles at A are equal, the unit arcs

into which their sides divide the arc CD are equal. Similarly for the arc FE. Sec. 203

4. $\therefore \dfrac{\text{arc } CD}{\text{arc } FE} = \dfrac{m}{r}.$

5. $\therefore \dfrac{\angle A}{\angle B} = \dfrac{\text{arc } CD}{\text{arc } FE}.$ Steps (2) and (4). Q.E.D.

292. Preparatory. Find where the following propositions have been proved:

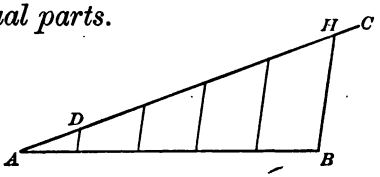
1. *If a line parallel to one side of a triangle bisects one of the two remaining sides, it bisects the other also.*

2. *If four parallels intercept equal segments upon one secant, they intercept equal segments on all secants.*

3. *In general, if n parallels intercept equal segments on one secant, they intercept equal segments on all secants.*

PROPOSITION IV. THEOREM

293. *To divide a given segment into any given number, n, of equal parts.*



Given the segment AB.

Required to divide AB into n equal parts.

Construction. 1. From A draw a line AC of sufficient length. On it lay off a convenient unit, AD, n times. (In the figure n is taken as 5.)

2. Connect the last end point, H, with B.

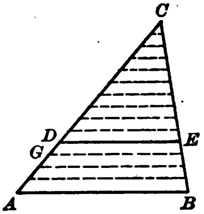
3. Through the other division points draw parallels to BH. Sec. 246

4. They will divide AB into n (5) equal parts. Q.E.F.

Supply the proof.

PROPOSITION V. THEOREM

294. *A line parallel to one of the sides of a triangle and cutting the other two sides, divides them into segments which are in the same ratio.*



Given $DE \parallel AB$.

To prove that $\frac{AD}{DC} = \frac{BE}{EC}$.

Proof. 1. Let DG be a common measure of AD and DC, and let it be contained m times in AD and r times in DC. Then, $\frac{AD}{DC} = \frac{m}{r}$.

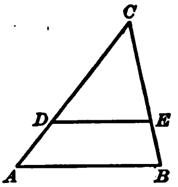
2. By applying DG repeatedly, divide AD and DC into m and r equal parts respectively.

3. From the points of division draw parallels to DE, extending them to intersect BC.

4. The parallels to DE therefore divide BE into m equal parts and EC into r equal parts. Hence, $\frac{BE}{EC} = \frac{m}{r}$. Sec. 163

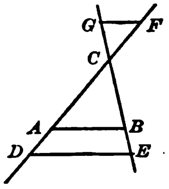
5. $\therefore \frac{AD}{DC} = \frac{BE}{EC}$. From (1) and (4). Q.E.D.

When AD and DC have no common measure, we see, as in Sec. 282, that their ratio can be found within any desired degree of approximation, and thus the demonstration above suffices in all cases.



295. Cor. 1. *If in the figure,* $\frac{AD}{DC}=\frac{BE}{EC}$, *then* $\frac{AC}{DC}=\frac{BC}{EC}$, *and conversely.*

Suggestion. Apply Sec. 289, VI and II.

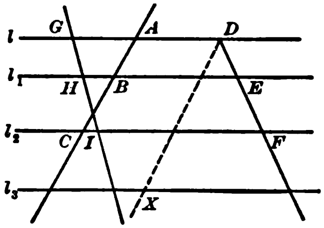


296. Cor. 2. *If in the figure, DE and FG are parallel to AB, show that*

$$\frac{AC}{CF}=\frac{BC}{CG}, \text{ also } \frac{DA}{AF}=\frac{EB}{BG}.$$

Suggestion. Use the method of the main proof.

297. Cor. 3. *The ratio of the segments cut from a secant by two or more parallels is the same for all positions of the secant.*



Suggestion. In the figure, l, l_1, l_2, l_3 are parallel and AC, DF and GI are different positions of a secant. Draw $DX \parallel AC$. Compare the segments of DX with those of AC and of DF.

EXERCISES

1. Draw a line 6 in. long. By careful construction divide it into 5 equal parts. Check your work by spanning one of the parts with compasses, and applying the length to each of the others.

2. In the figure of Cor. 1, given $AD = 4$ in., $DC = 10$ in., $CB = 12$ in. Find CE and EB.

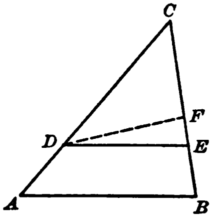
3. In the figure of Cor. 2, given $BE = 3$ in., $BC = 8$ in., $GC = 5$ in., and $AC = 10$ in. Find AD, CF and DF.

4. In the figure of Cor. 2, given $DF = 20$ in., $GE = 18$ in., $CF = 5$ in., $BE = 2$ in. Find AD, AC, BC and GC.

5. In the figure of Cor. 3, given $ED = 18$, $EF = 22$, $GI = 20$. Find GH and HI.

PROPOSITION VI. THEOREM

298. *If a line divides two sides of a triangle proportionally, it is parallel to the third side.*



Given $\frac{AD}{DC} = \frac{BE}{EC}$.

To prove that $DE \parallel AB$.

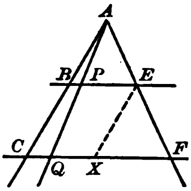
Proof. 1. If DE is not $\parallel AB$, let the parallel to AB through D meet BC in F. Then, $\frac{BF}{FC} = \frac{AD}{DC}$. Sec. 294

2. Hence, by the hypothesis, $\frac{BF}{FC} = \frac{BE}{EC}$. But this cannot be true, since the fraction on the left has a larger numerator than that on the right, and a smaller denominator, and is therefore larger than the fraction on the right.

3. If the point F fell between B and E, it would follow similarly that the resulting ratio would not equal $\frac{AD}{DC}$.

4. $\therefore DF$ coincides with DE and $DE \parallel AB$. Q.E.D.

299. COROLLARY. *The segments on two or more parallel straight lines cut out by the sides of an angle are proportional to the distances of the parallels from the vertex of the angle, measured on any line through the vertex.*



Given the parallels BE and CF cut by the sides of the $\angle A$.

To prove that $\frac{FC}{EB}=\frac{AC}{AB}=\frac{AQ}{AP}=\frac{AF}{AE}$.

Proof. 1. Draw $EX \parallel AC$.

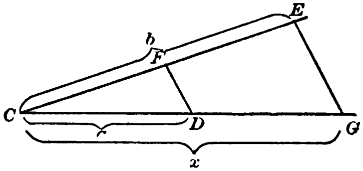
2. Then, $\frac{FC}{XC}$ or $\frac{FC}{EB}=\frac{FA}{EA}$, in $\triangle AFC$. Sec. 294

3. But $\frac{AF}{AE}=\frac{AC}{AB}=\frac{AQ}{AP}$. Sec. 294

4. $\therefore \frac{FC}{EB}=\frac{AC}{AB}=\frac{AQ}{AP}=\frac{AF}{AE}$. Sec. 78, I

PROPOSITION VII. PROBLEM

300. *To construct a segment whose length shall be the product of the lengths of two given segments, all measured in the same unit.*



Given two segments, a and b.

To find a segment x, such that $x = ab$.

Construction. On any two intersecting lines, lay off $CD = a$, $CE=b$, $CF=1$. Connect D, F. Draw $EG \parallel FD$. Then $CG=x$.

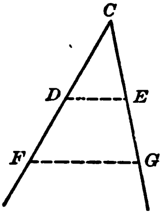
Proof. 1. $\frac{CG}{CD}=\frac{CE}{CF}$. Why?

2. $\therefore \frac{CG}{a}=\frac{b}{1}$. Given

3. $\therefore CG = ab$, or x, the desired segment. Sec. 289, I. Q.E.D.

NOTE. — In Proposition VII we have constructed a segment which represents the product of the numerical measures of two segments. It will be shown later that the number of square units in a rectangle whose sides are the given segments also is the product of the number of linear units in the segments.

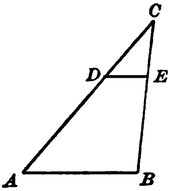
301. Corollary. *Similarly, a segment may be found whose length is the quotient of the lengths of two given segments all measured in the same unit.*



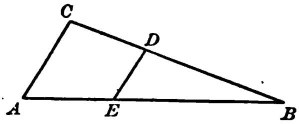
Suggestion. To find $x = \frac{a}{b}$, lay off $CD = a$, $CF = b$, $CG = 1$. Connect FG. Draw $DE \parallel FG$. Then $CE = x$.

EXERCISES

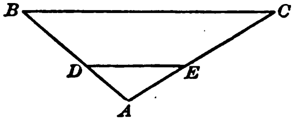
1. Given that $DE \parallel AB$, $CA = 18$, $CD = 8$, $CB = 15$, find CE.



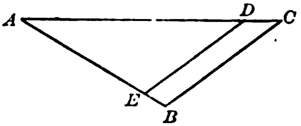
2. Given $DE \parallel CA$, $CD = 4$, $CB = 10$, $AE = 5$. Find AB.



3. Given $DE \parallel BC$, $DA = 16$, $DB = 20$, $AE = 18$. Find EC.



4. Given $ED \parallel BC$, $AE = 1.5$, $AB = 1.8$, $DC = .6$. Find AD.



5. If the mid-points of the sides of a parallelogram are connected in order, prove that the figure formed is a parallelogram.

6. Construct the segment $x = ab$, for the following values of a and b. In each case also calculate the value of x, and verify the approximate accuracy of your construction, by measurement of the segment x:

$a =$	2	3	4	1.5	.7	1.4
$b =$	3	4	5	2.5	.7	.9

7. Construct the segment $x = \frac{a}{b}$, and verify as above for the following values of a and b.

$a =$	6	15	12	7.2	7.2	25	2.5	2.5
$b =$	3	5	8	.6	6	5	5	.5

8. In Fig. 1, the short lines being parallel and equidistant apart, if BC were 1 in. long, how long would be the line at 9 parallel to BC? Also the lines at 1, 3, 5, 4, 8?

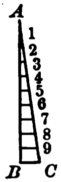


FIG. 1

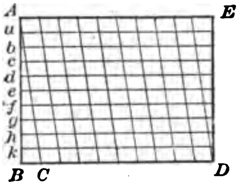


FIG. 2

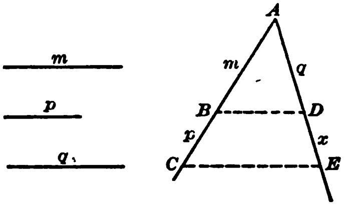
9. In Fig. 2, $BD = 1$ in., AE and BD are parallels. Each inch on BD and AE is divided into ten equal parts, and parallels are drawn as indicated. Since $BC = .1$ in., what is the length of the short line at k? At g? At d? At a?

10. Point out a line in Fig. 2 of Exercise 9 that is .09 in.; .19 in.; .74 in.

11. If in Fig. 2, BD is 5 in. long, explain how the scale may be used to find any number of inches and hundredths of an inch, up to 5 inches. Such a figure is called a *diagonal scale*, and when carefully made on steel or ivory will give lengths to a hundredth of an inch.

PROPOSITION VIII. PROBLEM

302. *To construct a fourth proportional to three given line segments.*



Given the segments m, p, q.

To find a fourth segment x, such that $\frac{m}{p}=\frac{q}{x}$.

Construction. The proportion at once suggests that the four segments may be placed so as to be the segments of two sides of a triangle made by a parallel to the base. Sec. 294

State the construction from the figure, and give the proof.

EXERCISES

1. Draw three arbitrary segments, a, b, c; by construction, find a segment, x, such that $\frac{a}{b}=\frac{c}{x}$.

Verify the approximate accuracy of your work by measuring a, b, c, calculating the value of x determined by these values, and comparing with the length of the segment x found by construction.

By construction find x such that:

2. $\frac{x}{5}=\frac{3}{7}$.

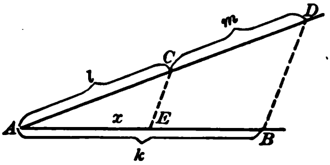
3. $\frac{x}{2}=\frac{4}{5}$.

4. $\frac{2}{9}=\frac{x}{4}$.

5. $\frac{3}{x}=\frac{2}{3}$.

PROPOSITION IX. PROBLEM

303. *To divide a given segment into parts proportional to two given segments.*



Given the segment, k, to be divided, and the other two segments, l and m.

To find a segment x such that $\frac{x}{k-x}=\frac{l}{m}$.

Construction. 1. On any two intersecting straight lines, lay off the segment k on one, and l and m on the other, as indicated in the figure.

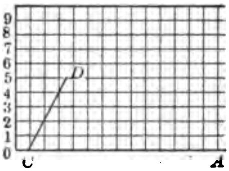
2. Draw DB, and through C draw $CE \parallel DB$. Then $AE=x$. Supply the proof.

304. COROLLARY. ***To divide a given segment into parts proportional to n given segments.***

Take a particular case but show that the method is general.

EXERCISES

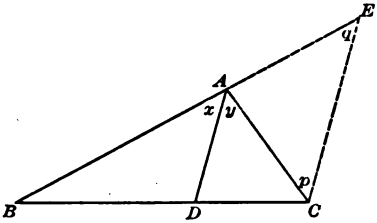
1. On squared paper, with C as the center and a radius equal to the line to be divided into parts, draw an arc cutting the fifth parallel to CA at D. Show that these parallels divide the line CD into five equal parts.



2. How would you use the method of Exercise 1 to divide a line of length CD into four equal parts? Into six equal parts?

PROPOSITION X. THEOREM

305. *The bisector of an angle of a triangle divides the opposite side into segments proportional to the adjacent sides.*



Given $\triangle ABC$ in which AD bisects $\angle A$.

To prove that $\frac{AB}{AC}=\frac{BD}{DC}$.

Proof. 1. Through C draw a line $\parallel AD$, and produce BA to meet it at E.

2. Then, $\angle x = \angle q$ and $\angle p = \angle y$. Why?

3. $\therefore \angle p = \angle x = \angle q$. Why?

4. $\therefore AC = AE$. Why?

5. But $\frac{BA}{AE}=\frac{BD}{DC}$. Sec. 294

6. $\therefore \frac{AB}{AC}=\frac{BD}{DC}$. Step (4). Q.E.D.

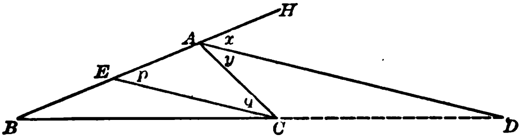
EXERCISES

1. The sides of a triangle are 24, 30, 36 in. Calculate the lengths of the segments into which the bisectors of the three angles divide the sides.

2. The sides of a triangle are $5a$, $6a$, $8a$. Calculate the segments of the sides as in Exercise 1.

3. The sides of a triangle are a, b, c. Calculate their segments as in Exercise 1.

306. COROLLARY. *The bisector of the exterior angle of a triangle determines, on the opposite side extended, segments proportional to the adjacent sides.*



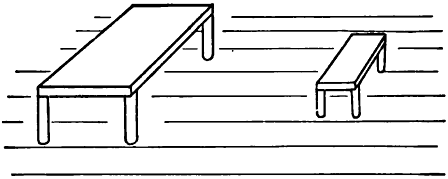
For, in the figure, AD bisects the exterior angle CAH, meeting BC extended at D, and $CE \parallel DA$.

$\triangle ECA$ is isosceles and $EA = AC$.

Then, $$\frac{BA}{EA} = \frac{BD}{CD} = \frac{BA}{AC}.$$ Why?

The point D does not actually divide the segment BC, but D is a point on the segment BC extended such that the segments DC and DB satisfy Prop. X.

307. Preparatory.



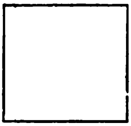
These two tables have the same *shape*, but not the same *size;* the dimensions of the one are twice those of the other. There is then a marked similarity in appearance. Polygons that have the same shape are called "similar," in accordance with the following formal definition.

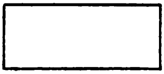
308. Similar Polygons. Polygons that have their corresponding angles equal and their corresponding sides proportional are called **similar polygons.**

309. Ratio of Similitude. The constant ratio between the corresponding sides of similar polygons is called the **ratio of similitude** of the polygons.

EXERCISES

1. Are these figures similar? If not, why not?





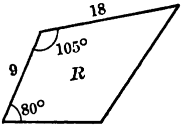
2. Are two equilateral triangles similar? Why?

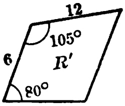
3. Are two isosceles triangles similar?

4. What two things must be proved about two figures in order to show that they are similar?

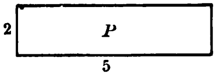
5. Two equilateral triangles have sides respectively 3 in. and 9 in. long. What is their ratio of similitude?

6. What must be proved in addition to what is given in the figure to show that *R* and *R'* are similar?





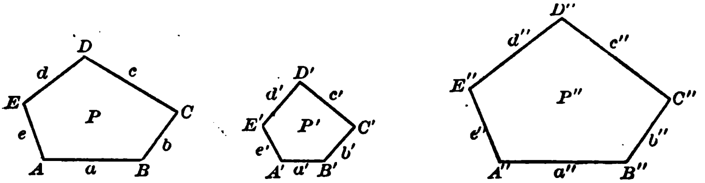
7. State the dimensions of a rectangle similar to *P*.



8. What is the ratio of similitude of the polygon of Ex. 7 and the one taken similar to it?

PROPOSITION XI. THEOREM

310. *If two polygons are similar to a third polygon, they are similar to each other.*



Given P' similar to P, and P'' similar to P.

To prove that P' is similar to P''.

To do this we must show:

(I) $\angle A' = \angle A''$, $\angle B' = \angle B''$, etc.

(II) $\frac{b'}{b''} = \frac{a'}{a''}$; $\frac{c'}{c''} = \frac{a'}{a''}$, etc.

Proof. 1. $\angle A' = \angle A''$, $\angle B' = \angle B''$, etc. Sec. 78, I
This proves (I)

2. $\frac{b}{b'} = \frac{a}{a'}$; $\frac{c}{c'} = \frac{a}{a'}$, etc.

and, $\frac{b}{b''} = \frac{a}{a''}$; $\frac{c}{a''} = \frac{a}{a''}$, etc. Given, Sec. 309

3. In step 2, dividing the equations of the second set, member for member, by the corresponding equations of the first set, we obtain:

$\frac{b'}{b''} = \frac{a'}{a''}$; $\frac{c'}{c''} = \frac{a'}{a''}$, etc. This proves (II)

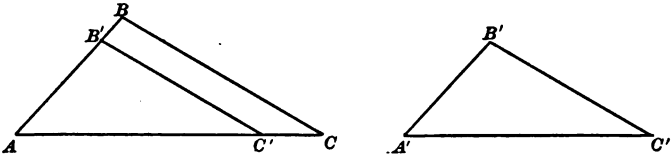
4. $\therefore$ P' is similar to P''. Sec. 308. Q.E.D.

EXERCISE

Show that all squares are similar polygons.

PROPOSITION XII. THEOREM

311. *If the three angles of a triangle are respectively equal to the three angles of another triangle, the triangles are similar.*



Given $\triangle$s ABC and $A'B'C'$, with $\angle A = \angle A'$, $\angle B = \angle B'$, and $\angle C = \angle C'$.

To prove that $\dfrac{AB}{A'B'} = \dfrac{BC}{B'C'} = \dfrac{AC}{A'C'}$.

Proof. 1. Place $\triangle A'B'C'$ on $\triangle ABC$ so that $\angle A'$ coincides with $\angle A$.

2. Then $B'C' \parallel BC$, since $\angle B' = \angle B$. Why?

3. $\therefore \dfrac{AB}{AB'} = \dfrac{AC}{AC'}$, or $\dfrac{AB}{A'B'} = \dfrac{AC}{A'C'}$. Why?

4. Also, $\dfrac{AB}{AB'} = \dfrac{BC}{B'C'}$, or $\dfrac{AB}{A'B'} = \dfrac{BC}{B'C'}$. Sec. 294

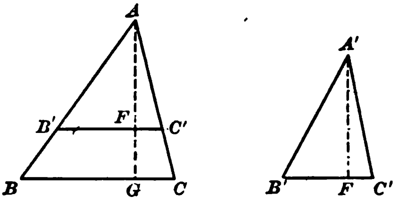
5. $\therefore \dfrac{AB}{A'B'} = \dfrac{AC}{A'C'} = \dfrac{BC}{B'C'}$. From (3) and (4)

6. $\therefore$ $\triangle$s ABC and $A'B'C'$ are similar. Sec. 308. Q.E.D.

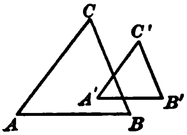
NOTE. — Thus, in the case of triangles, the condition of equality of corresponding angles is sufficient to establish similarity.

312. COR. 1. *If two triangles have two angles of one respectively equal to two angles of the other, the triangles are similar.*

313. Cor. 2. *If two triangles are similar, the ratio of the pair of altitudes from the corresponding vertices equals the ratio of similarity of the triangles.*

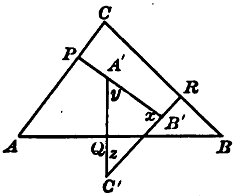


Suggestion. Place the triangles so that the equal angles *A* and *A'* coincide. Then *B'C'* is parallel to *BC*, and triangles *AB'F* and *ABG* are similar.



314. Cor. 3. *If two triangles have their corresponding sides respectively parallel, the triangles are similar.*

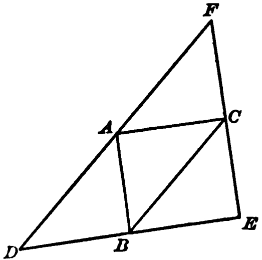
Apply Sec. 141.



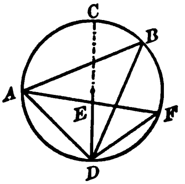
315. Cor. 4. *Two triangles are similar if their corresponding sides are respectively perpendicular.*

Apply Sec. 142.

EXERCISES



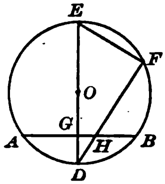
1. The sides of the triangle *DEF* are respectively parallel to the corresponding sides of *ABC*. Read all the triangles of the figure that are similar to *ABC*.



2. In the figure, given that $\angle DEF = \angle DAB$. Show that $\triangle DEF$ is similar to $\triangle DAB$. Hence, show that arc $AC =$ arc *BF*.

3. If two triangles are similar, show that the ratio of the bisectors of two corresponding angles equals the ratio of similitude of the triangles.

4. Prove that the ratio of the medians to any two corresponding sides of similar triangles equals the ratio of similitude of the triangles.

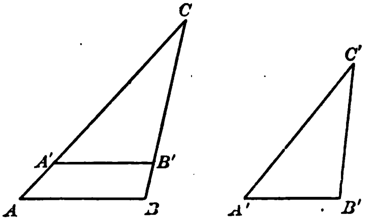


5. Every parallel to a side of a triangle is bisected by the median to that side.

6. If DE is perpendicular to chord AB, show that triangles GHD and DEF are similar. Hence, show that $DG \cdot DE = DH \cdot DF$.

PROPOSITION XIII. THEOREM

316. *If two triangles have an angle of one equal to an angle of the other and the sides including these angles proportional, the triangles are similar.*



Given $\triangle$ ABC and $A'B'C'$, in which $\angle C = \angle C'$, and

$$\frac{AC}{A'C'} = \frac{BC}{B'C'}.$$

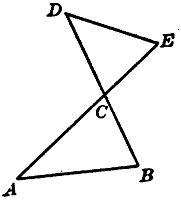
To prove that $\triangle$ ABC and $A'B'C'$ are similar.

Proof. 1. Place $\triangle A'B'C'$ on $\triangle ABC$ so that $\angle C'$ coincides with $\angle C$.

Then, $\frac{A'A}{A'C} = \frac{B'B}{B'C}$. Sec. 289, VII and II

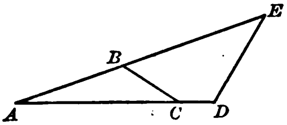
2. $\therefore$ $A'B'$ is parallel to AB. Why?
3. $\therefore$ $\angle A = \angle A'$ and $\angle B = \angle B'$. Why?
4. $\therefore$ the triangles are similar. Why?

EXERCISES



1. Given that $\frac{CE}{CD} = \frac{CB}{CA}$, show that triangles ABC and DCE are similar.

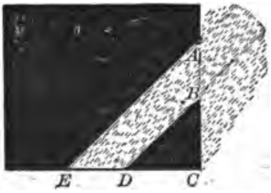
2. In the figure of Ex. 1, given that $AB = 12$, $BC = 9$, $DE = 8$, and $CE = 6$. Show that the $\triangle$ ABC and DCE are similar. Given further that $AC = 10$; find DC.



3. If $AB = 6$, $AC = 8$, $AD = 10$, what must be the length of AE in order that triangles ABC and ADE may be similar?

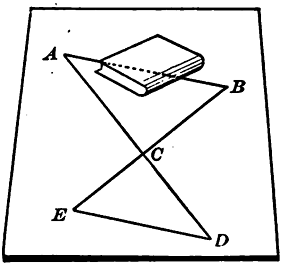
4. In the figure of Ex. 3, given that triangles ABC and ADE are similar, that ED is half of AD and that $AB = 8$. Find BC.

5. In the figure of Ex. 3, given that triangles ABC and ADE are similar, and that $AE = 20$, $AD = 15$, $DE = 10$, $BC = 8$. Find AB, AC, AD.



6. Show how to determine the height of the window, AB, by measuring BC, CD and the length DE of the portion of the floor illuminated by sunshine through the window.

7. Place a book on a sheet of paper, and mark two points A and B so that the line connecting them passes under the book. Select a point C, such that the lines AC and BC do not pass under the book. Measure AC and BC and prolong each by $\frac{3}{4}$ of its length. Measure DE and calculate AB.

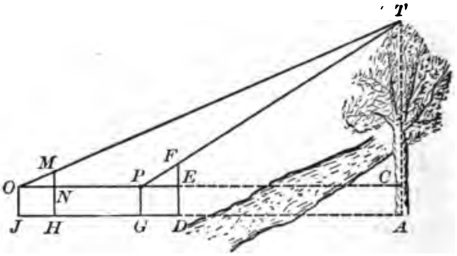


Verify the work by removing the book and measuring AB.

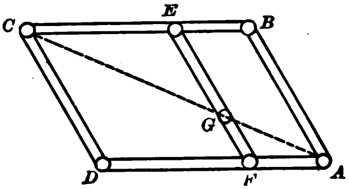
8. To measure the height AT of the tree on the other side of the river, the observer PG of height 5.8 ft. set up the staff FD of height 9 ft. in two positions D and H in line with the tree, and placed himself in turn in the positions G and J so that he could just see the top of the tree. By measurement he found the distance $GD = 8$ ft., $JH = 10.2$ ft., $HG = 20$ ft. Find the height of the tree and the distance DA.

Suggestion. Letting $TC = x$, and $EC = y$, we have from $\frac{PE}{EF} = \frac{PC}{CT}$, $\frac{8}{3.2} = \frac{8 + y}{x}$.

Similarly, a second equation for x and y can be written from $\triangle TOC$.

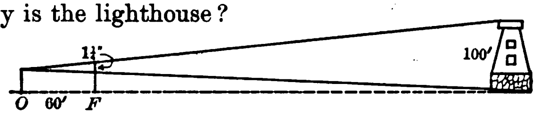


9. Enlargements of drawings can be made with an instrument like that indicated in the figure. $ABCD$ is a jointed parallelogram with fixed pivot at A. EF is a movable crossbar placed parallel to CD, so that the ratio $\frac{AD}{AF}$ shall have



a desired value, r. G is a tracing point, inserted on EF in line with A and C in any selected position. Show that it remains in line with A and C in all positions of the instrument and that if G traces any figure, C will describe that figure enlarged r times.

10. To an officer of a ship looking through his spy glass a strip $1\frac{1}{4}$ in. wide on a flagstaff 60 ft. away seems to measure the height of a lighthouse that he knows is 100 ft. high. How far away is the lighthouse?

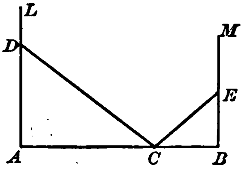


11. A post 4 ft. higher than an observer's eye and 6 ft. from him appears to be just as high as a barn 48 ft. away. How high is the barn?

12. On a straight line AB, a point C is arbitrarily selected; then on the perpendiculars AL and BM, distances are laid off such that $AD \cdot BE = AC \cdot BC$. Show that the angle DCE is a right angle.

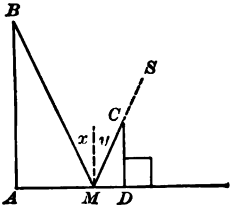
Suggestion. The given relation can be written $\frac{AD}{AC} = \frac{BC}{BE}$.

$\therefore$ △ ADC, BCE are similar.



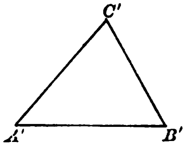
13. Show how to measure the height AB of the wall of a room by placing a chair CD, so that when sighting across the back of the chair, the image of B is seen at a fixed point, M, in the mirror. A ray of light is reflected so as to make $\angle x = \angle y$. (Ex. 14, p. 70.) What three lines must be measured in order to compute AB?

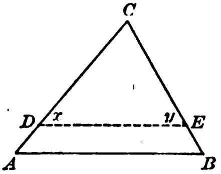
NOTE. — For convenience in measuring the lines, the point M may be marked by a small piece of paper. The work may be checked by repeating the measurements for a different position of the mirror.



PROPOSITION XIV. THEOREM

317. *If two triangles have their sides respectively proportional, the triangles are similar.*





Given △ ABC and $A'B'C'$ in which $\frac{AC}{A'C'}=\frac{BC}{B'C'}=\frac{AB}{A'B'}$.

To prove that $\angle A=\angle A'$, $\angle B=\angle B'$, $\angle C=\angle C'$.

Proof. 1. On the sides of $\triangle ABC$ lay off $CD=C'A'$ and $CE=C'B'$.

2. Since $\frac{CD}{CA}=\frac{CE}{CB}$, Given

then, $\triangle CDE$ is similar to $\triangle ABC$. Sec. 316

3. $\therefore \frac{CD}{CA}=\frac{DE}{AB}$. Secs. 298, 299

4. $\frac{C'A'}{CA}=\frac{DE}{AB}$. Step (1) and Sec. 78, I

5. But $\frac{C'A'}{CA}=\frac{A'B'}{AB}$. Given

6. $\therefore \frac{DE}{AB}=\frac{A'B'}{AB}$. From (4) and (5)

7. $\therefore DE=A'B'$.

8. $\therefore \triangle CDE$ is congruent to $\triangle A'B'C'$. Why?

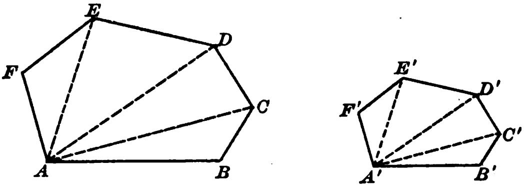
9. $\therefore \triangle A'B'C'$ is similar to $\triangle ABC$. Sec. 308. Q.E.D.

318. Diagonals connecting corresponding vertices of polygons are called **corresponding diagonals.**

319. If two polygons are divided into triangles by drawing diagonals, two triangles, one from each polygon, are said to be **correspondingly placed**, if the sides of the one are the lines corresponding to the sides of the other.

PROPOSITION XV. THEOREM

320. *If two similar polygons are divided into triangles by diagonals drawn from corresponding vertices, the correspondingly placed triangles are similar.*



Given two similar polygons ABC and $A'B'C'$.

To prove that $\triangle\ ABC$ and $\triangle\ A'B'C'$ are similar; also $\triangle\ ACD$ and $A'C'D'$, and so on.

Proof. 1. $\angle B = \angle B'$, Why?

2. and $\frac{AB}{BC} = \frac{A'B'}{B'C'}$. Why?

3. $\therefore$ $\triangle$ ABC and $A'B'C'$ are similar. Sec. 316

4. In $\triangle$ ACD and $A'C'D'$, $\angle ACD = \angle A'C'D'$, because $\angle C = \angle C'$, and $\angle ACB = \angle A'C'B'$. Step (3)

5. Further, from $\triangle$ ABC and $A'B'C'$, $\frac{AC}{A'C'} = \frac{BC}{B'C'}$.

6. But, $\frac{BC}{B'C'} = \frac{CD}{C'D'}$. Given

$\therefore$ $\frac{AC}{A'C'} = \frac{CD}{C'D'}$.

7 $\therefore$ $\triangle$ ACD and $A'C'D'$ are similar.

In the same way, the other triangles are seen to be similar and correspondingly placed.

321. Corollary. *Conversely, if the diagonals drawn from one vertex in each of two polygons of the same number of sides divide them into triangles, such that all pairs of correspondingly placed triangles are similar, then the polygons are similar.*

Proposition XVI. Theorem

322. *The perimeters of two similar polygons are in the same ratio as any two corresponding sides.*

Given two similar polygons, having the sides a, b, c, $\cdots$ and a', b', c', $\cdots$ respectively.

To prove that $\dfrac{a+b+c+\cdots}{a'+b'+c'+\cdots}=\dfrac{a}{a'}$, or $\dfrac{b}{b'}$, and so on.

Proof. 1. Since the figures are similar, $\dfrac{b}{b'}=\dfrac{a}{a'}$; $\dfrac{c}{c'}=\dfrac{a}{a'}$; $\dfrac{d}{d'}=\dfrac{a}{a'}$; etc.

2. $\therefore \dfrac{a+b+c+\cdots}{a'+b'+c'+\cdots}=\dfrac{a}{a'}$, or $\dfrac{b}{b'}$, and so on. Sec. 290

3. $\therefore$ the perimeters have the same ratio as any two corresponding sides.

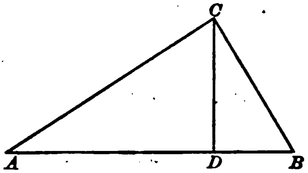
Proposition XVII. Theorem

323. *If in a right triangle, a perpendicular is drawn from the vertex of the right angle to the hypotenuse, then:*

(I) *The two triangles thus formed are similar to each other, and to the whole triangle.*

(II) *The perpendicular is a mean proportional between the segments of the hypotenuse.*

(III) *Each side of the right triangle is a mean proportional between the hypotenuse and the segment adjacent to that side.*



Given △ ABC, with right angle at C, and $CD \perp AB$.

(I) **To prove** that $\triangle$s ADC and CDB are similar to △ ABC and to each other.

Proof. 1. △ ADC is similar to △ ABC, for they both have a right angle, and ∠ A is common. Sec. 311

2. △ DBC is similar to △ ABC for they are right triangles and have ∠ B in common. Why?

3. ∴ △ ADC is similar to △ CDB. Sec. 310

(II) **To prove** that $\frac{AD}{DC}=\frac{DC}{DB}$, or that $\overline{DC}^2 = AD \cdot DB$.

Proof. $\frac{AD}{DC}$ and $\frac{DC}{DB}$ are equal, since they are ratios of corresponding sides in similar triangles. (Name them.) Sec. 308

(III) **To prove** that $\frac{AB}{AC}=\frac{AC}{AD}$ and $\frac{AB}{BC}=\frac{BC}{BD}$, or that $\overline{AC}^2 = AB \cdot AD$ and $\overline{BC}^2 = AB \cdot BD$.

Proof. $\frac{AB}{AC}$ and $\frac{AC}{AD}$ are the ratios of corresponding sides in the similar triangles ABC and ACD, and therefore equal.

Similarly prove the other equation.

324. COR. 1. *The perpendicular from any point on a circle to a diameter is a mean proportional between the segments of the diameter.*

325. Cor. 2. *If a perpendicular is drawn from any point of a circle to a diameter, the chord from that point to either end of the diameter is a mean proportional between the diameter and the segment adjacent to the chord.*

326. Cor. 3. *The squares on the two sides of a right triangle are proportional to the segments of the hypotenuse adjacent to those sides.*

For, $\overline{AC}^2 = AB \cdot AD$, and $\overline{BC}^2 = AB \cdot BD$. Case III

$$\therefore \frac{\overline{AC}^2}{\overline{BC}^2} = \frac{AB \cdot AD}{AB \cdot BD} = \frac{AD}{BD}.$$

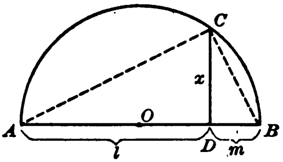
327. Cor. 4. *The square on the hypotenuse and the square on either side of a right triangle are proportional to the hypotenuse and the segment of the hypotenuse adjacent to that side.*

For, $\overline{AC}^2 = AB \cdot AD$. Case III

Dividing by $\overline{AB}^2$, $\dfrac{\overline{AC}^2}{\overline{AB}^2} = \dfrac{AB \cdot AD}{AB \cdot AB} = \dfrac{AD}{AB}$, or, $\dfrac{\overline{AB}^2}{\overline{AC}^2} = \dfrac{AB}{AD}$.

Proposition XVIII. Problem

328. *To construct the mean proportional between two given straight lines.*



Given the segments l and m.

To construct a line x such that $lm = x^2$.

Construction. 1. Construct $AB = l + m$.

2. Erect the perpendicular DC, and construct a right triangle on AB with the vertex of the right angle on DC. Sec. 233

Proof. Apply the relation stated in Sec. 323, Case II.

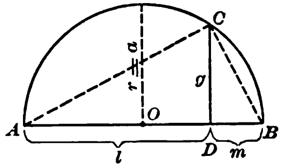
329. Geometric Mean. The mean proportional between two magnitudes is called their **geometric mean.**

330. Arithmetical Mean. The average of two magnitudes (that is, one half their sum) is called their **arithmetical mean.**

EXERCISE

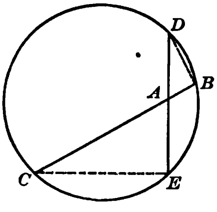
Show that the arithmetical mean between two unequal numbers is greater than their geometric mean.

Suggestion. Let the numbers be represented by line segments l and m, their arithmetical mean by a, and their geometric mean by g. In the figure compare g with a.



PROPOSITION XIX. THEOREM

331. *The product of the segments of a chord that passes through a fixed point within a circle is constant, that is, the same for all such chords.*



Given the chords DAE and BAC.

To prove that $AB \cdot AC = DA \cdot AE$.

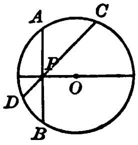
Proof. 1. Draw DB and CE, then triangles ABD and ACE are similar. Apply Secs. 234 and 311

2. $\therefore \dfrac{AB}{AD} = \dfrac{AE}{AC}$, or $AB \cdot AC = DA \cdot AE$. Sec. 308. Q.E.D.

EXERCISES

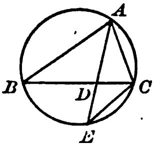
1. If a quadrilateral is inscriptible, show that each diagonal divides the other into two segments, whose product is the same for both.

2. Prove that the product of the segments of any chord through a point in a circle equals the square on one half of the chord through the same point perpendicular to the diameter through that point.

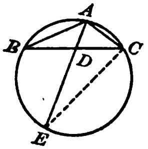


Suggestion. Apply Secs. 324, 210 and 331.

3. If in the triangle ABC, the bisector AD of the angle BAC is produced to meet the circumscribed circle at E, show that triangles ABD and ACE are similar.



4. The square of the bisector of an angle of a triangle is equal to the product of the sides including this angle less the product of the segments into which the bisector divides the third side.



Suggestion. Let DA bisect $\angle BAC$.

To show that $\overline{AD}^2 = AB \cdot AC - BD \cdot DC$.

1. Draw the circumscribed circle and produce AD to E, then

$$BD \cdot DC = AD \cdot DE. \qquad \text{Sec. 331}$$

2. But,

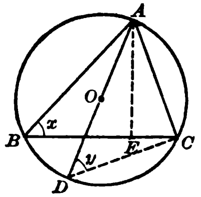
$$\frac{AB}{AE} = \frac{AD}{AC}. \qquad \text{Exc. 3}$$

or,

$$\begin{aligned} AB \cdot AC &= AE \cdot AD. \\ &= AD(AD + DE) \\ &= \overline{AD}^2 + AD \cdot DE \\ &= \overline{AD}^2 + BD \cdot DC. \qquad \text{Step (1)} \end{aligned}$$

5. By using the proposition of Sec. 305, and the result of Exercise 4, calculate the lengths of the bisectors of the angles of a triangle whose sides are 7, 11, 12.

6. The product of any two sides of a triangle equals the product of the altitude on the third side by the diameter of the circumscribed circle.



Suggestion. To show that $AB \cdot AC = AD \cdot AE$. Observe that $\angle x = \angle y$.

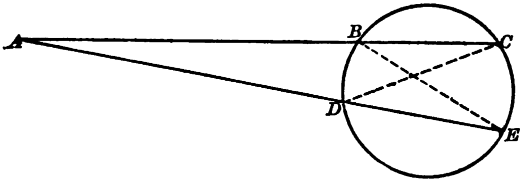
$$\angle ACD = 90^\circ = \angle AEB.$$

$\therefore \triangle ABE$ is similar to $\triangle ACD$.

7. An altitude of a triangle is 12. The including sides are 18 and 30. Find the diameter of the circumscribed circle.

Proposition XX. Theorem

332. *If from a fixed point without a circle a secant is drawn, the product of the whole secant and the external segment is the same for all such secants.*



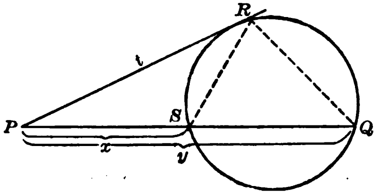
Given the two secants ABC and ADE.

To prove that $AB \cdot AC = AD \cdot AE$.

Proof. 1. $\triangle$s ABE and ADC are similar. Sec. 311

2. Complete the proof.

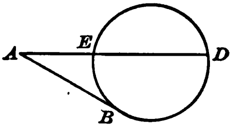
333. Corollary. *If a tangent and a secant are drawn from the same point to a circle, the tangent is a mean proportional between the whole secant and its external segment.*



Suggestion. In the figure, show that $t^2 = xy$.

EXERCISES

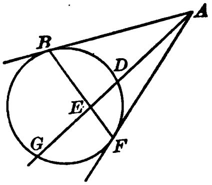
1. Given $AE = 3$, $AD = 12$, find AB.



2. In the figure of Exercise 1, given $AB = 7$, $AE = 3$, find DE.

3. As in Exercise 2, given $AB = 6$, $ED = 5$, find AE.

4. Given $AB = 9$, $AG = 12$, $ED = 2$, $BF = 6$, find AF, AD, EG, BE, EF. Apply Secs. 333 and 331.



5. Show how to construct the figure of Exercise 4 from the data given and found.

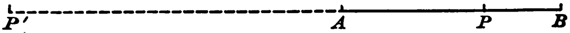
6. Prove that two isosceles triangles are similar if any angle of one is equal to the corresponding angle of the other.

7. Prove that the diagonals of a trapezoid divide each other proportionally.

8. Regarding 30 as 5×6, construct a line x according to Prop. XVII, such that $x^2 = 30$, or $x = \sqrt{30}$. Also construct $\sqrt{30}$ by regarding 30 as 3×10. Compare the results.

9. A tangent from a point to a circle is 15 in. long; the point is 9 in. from the nearest point of the circle; find the radius.

334. Internal and External Division of Segments. The point P between the points A and B is said to divide the segment AB *internally* into the segments PA and PB, while P' on the prolongation of AB is said to divide it *externally* into the segments $P'A$ and $P'B$.

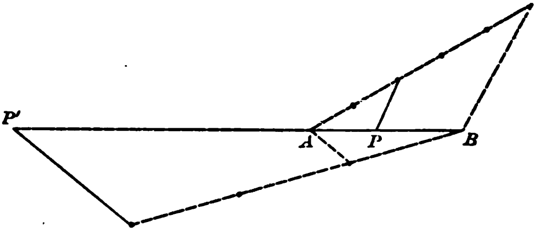


In either case the lengths of the segments are the distances from the point of division to the ends of the given line. If the line is divided internally, the sum of the segments equals the given line; and if the line is divided externally, the difference of the segments equals the line.

335. By applying Sec. 303, a line may be divided in a given ratio both internally and externally.

Let the segment be AB, and the ratio 2 to 3. Then in the figure

$$\frac{PA}{PB}=\frac{P'A}{P'B}.$$



336. A line divided internally and externally in the same ratio is said to be divided *harmonically.*

Thus, the line AB is divided *harmonically* at P and P' in the ratio 2 : 3.

337. Extreme and Mean Ratio. If a given line segment is divided into two unequal parts such that the greater of the parts is a mean proportional between the whole segment and the smaller part, the given segment is said to be divided in **extreme and mean ratio.**

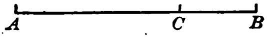


FIG. 1

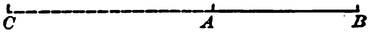


FIG. 2

If, in either Fig. 1 or Fig. 2 $\frac{AB}{AC}=\frac{AC}{CB}$, AB is divided at C in extreme and mean ratio. If the point C lies on AB extended, as in Fig. 2, AB is said to be divided in *extreme and mean ratio externally.*

The proportion may also be written $\overline{AC}^2 = AB \cdot BC$. Sec. 287

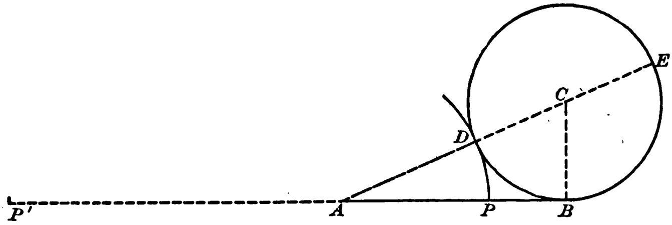
This method of division is often called **Golden Section**, and is one of the most prevalent proportions in nature and in art.

PROPOSITION XXI. PROBLEM

338. *To divide a given line segment in extreme and mean ratio.*

Given the segment AB.

To find a point P, such that $\frac{AB}{AP}=\frac{AP}{PB}$.



Construction. 1. Construct a circle tangent to AB at B with a radius equal to $\frac{1}{2}$ AB. Sec. 250

2. Draw AC and produce it to cut the circle at E.

3. With A as a center and AD as a radius, draw an arc cutting AB in P.

4. P divides AB in extreme and mean ratio.

Proof. 1. $\frac{AE}{AB}=\frac{AB}{AD}$. Sec. 333

2. $\frac{AE-AB}{AB}=\frac{AB-AD}{AD}$. Sec. 289, VII.

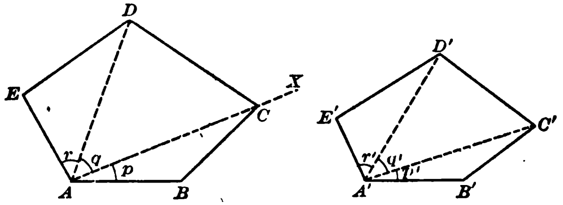
3. $\frac{AE-DE}{AB}=\frac{AB-AP}{AD}$. Sec. 78, I

4. $\frac{AP}{AB}=\frac{PB}{AP}$. Sec. 78, I

or, $\frac{AB}{AP}=\frac{AP}{PB}$. Sec. 289, IX. Q.E.D.

PROPOSITION XXII. PROBLEM

339. *To construct upon a line segment as a given side a polygon similar to a given polygon.*



Given line AB and polygon $A'B'C'\text{–}E'$.

To construct on AB the polygon $ABC\text{–}E$ similar to $A'B'C'\text{–}E'$.

Construction. 1. Draw the diagonals $A'C'$ and $A'D'$ in the given polygon.

2. At A draw $\angle p = \angle p'$, with side AB. Sec. 245

3. At B draw a line making an angle with AB equal to B' and meeting AX at C.

4. At A draw a line making $\angle q = \angle q'$.

5. At C draw a line CD making $\angle ACD = \angle A'C'D'$; where the lines (4) and (5) meet is the vertex D.

6. Proceed in the same way from A and D.

7. The figure $ABCDE$ is the required polygon.

Proof. △ ABC and $A'B'C'$, △ ACD and $A'C'D'$ and △ ADE and $A'D'E'$ are similar. ∴ $ABCDE$ is similar to $A'B'C'D'E'$. Sec. 321. Q.E.D.

EXERCISE

Prove that the bisector of the interior angle and the bisector of the exterior angle at the same vertex of a triangle divide the opposite side harmonically.

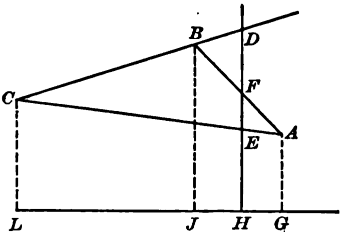
REVIEW EXERCISES

1. When three sides of a triangle (produced when necessary) are cut by a transversal, the product of one set of three segments not having a common extremity is equal to the product of the other set of such segments.

Suggestion. Let ABC be the given triangle, and EFD be the transversal.

To show that $AE \cdot CD \cdot BF = EC \cdot BD \cdot AF$, or,

$$\frac{AE}{EC} \cdot \frac{CD}{BD} \cdot \frac{BF}{AF} = 1.$$



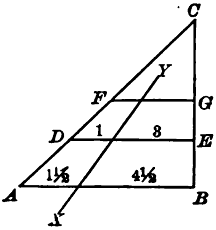
Drop perpendiculars upon any line perpendicular to the transversal.

Then, $\frac{AE}{EC} = \frac{GH}{HL}$. Sec. 297

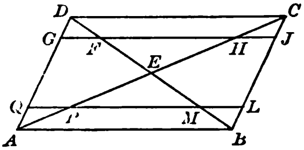
$\frac{CD}{BD} = \frac{HL}{JH}$.

$\frac{BF}{AF} = \frac{HJ}{GH}$.

Take the product of these ratios.



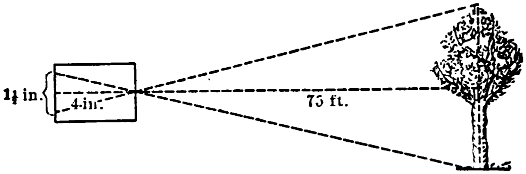
2. The lines AB and DE are divided as indicated by the numbers in the figure. Show that the line XY produced will pass through C. In what ratio is FG divided?



3. In the parallelogram $ABCD$, GJ and QL are parallel to AB. Read all the triangles that are similar to PME; to DGF; to APQ.

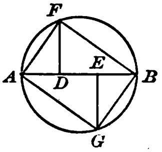
4. If light passes through a very small hole into a dark box, a fairly good image of the object from which the light comes is produced without a lens. By applying this principle, cheap photographic cameras, known as "pin-hole" cameras, have been constructed. If the plate in a pin-hole camera is

4 in. from the aperture, and if the picture of a tree 75 ft. away is $1\frac{1}{2}$ in. high, how high is the tree?



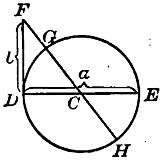
5. If the diameter of the moon is approximately 1000 mi., and the distance of the moon from the earth 240,000 mi., what is the diameter of a disk which when held 36 in. from the eye will just cover the moon? Express the result to nearest hundredth of an inch.

6. It is shown in studies on the strength of materials that the strongest rectangular beam that can be cut from a given circular log will have its breadth, b, so related to its height, h, that $\frac{b}{h}=\frac{1}{\sqrt{2}}$. Show that if perpendiculars be erected at the trisection points of a diameter AB, as indicated in the figure, then $AFBG$ will be a rectangle and $\frac{AF}{AG}=\frac{1}{\sqrt{2}}$. That is, if the log be cut along $AFBG$, the strongest possible rectangular beam will be obtained.



7. To construct two segments, knowing their difference a and their mean proportional b.

Suggestion. On a as a diameter draw a circle, and draw $FD \perp a$ and equal to b. Then, FH and FG are the two desired segments.



8. Let x and y be the segments required in Exercise 7; find them by solving the following equations for x and y:

$$x-y=a,\quad xy=b^2.$$

9. According to Exercise 7, find, by careful drawing, the two segments when $a = 6$, $b = 4$. Measure the segments and compare the values with those determined by the equations in Exercise 8.

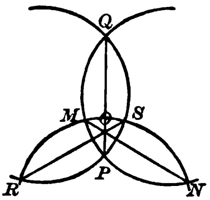
10. In a circle of diameter 16, a certain chord is cut by the diameter into segments 6 and 8. How far from the extremity of the diameter does the chord cut it?

11. The tangents to two intersecting circles, drawn from any point in the extension of their common chord, are equal.

Suggestion. Apply Sec. 333.

12. The common chords of three circles, all intersecting one another, are concurrent.

Suggestion. Let chords MN, RS intersect at O. Assume that QO does not meet the two circles at P, but in two distinct points, T, V (not indicated in the figure). Apply Sec. 331 to show that this supposition is incorrect.



13. Two circles are tangent internally. Prove that all chords of the larger circle drawn from the point of contact are divided proportionally by the smaller circle.

14. Two circles are tangent at P. Through P three lines pass, meeting one circle in points XYZ and the other circle in points $X'Y'Z'$. Prove that $\triangle XYZ$ is similar to $\triangle X'Y'Z'$.

15. The ratio of one of the equal sides of an isosceles triangle to the base is 2, and the perimeter is 25. Find the length of each side.

16. Two similar triangles have a pair of corresponding sides measuring 10 in. and 15 in. respectively. The altitude of the smaller triangle is 8 in. What is the altitude of the other triangle?

17. Find by geometric construction and by calculation the mean proportional between 2 in. and $4\frac{1}{2}$ in.

18. Find by geometric construction and by calculation the fourth proportional to .5 in., 2 in., and .25 in.

19. Divide a line 8 in. long into extreme and mean ratio. Measure the two segments and find their lengths to within one sixteenth of an inch. Verify by letting x equal the larger segment and solving the equation $x^2 = 8(8 - x)$.

20. a and b are two given lines and $x = \sqrt{ab}$. Construct a line equal to x.

21. Given $x = \sqrt{2\,ab}$, in which a and b are two line segments. Construct a line equal to x.

22. A chord 5 in. long is extended 4 in. beyond the circle. Find the length of the tangent drawn from the extremity of this line.

23. Through an external point to draw a secant to a circle so that the square of the intercepted chord shall equal the product of the segments of the secant.

24. To draw a line through an intersection of two circles so that the two chords thus formed shall have a given ratio.

25. Show how to find the center of a circular disk by means of a carpenter's square.

26. To inscribe in a given circle a triangle similar to a given triangle.

27. If the line joining the mid-points of the parallel sides of a trapezoid is extended and also the other two sides, then these lines are concurrent.

28. If two circles are tangent, the portion of their common external tangent included between the points of tangency is a mean proportional between their diameters.

Suggestion. Let C and C' be the centers, r and r' the radii of the two circles, A and A' the points of contact of the common tangent. Suppose $r' > r$, and draw $CB \perp C'A'$. Then:

$$\overline{AA'}^2 = \overline{CB}^2 = (r' + r)^2 - (r' - r)^2 = (2\,r)(2\,r').$$

29. C is the mid-point of the arc ACB subtended by the chord AB. Then CE is any other chord cutting the chord AB at the point D. Prove that $AC : CE = CD : AC$.

30. From any point on the arc subtended by a given chord, construct a chord that shall be bisected by the given chord.

Suggestion. Let C be the center of the circle, AB the given chord, D the given point, DE the desired chord and F its mid-point. Then F lies on AB, and FC is $\perp DE$. Hence F is the point where the semicircle erected on CD cuts AB.

HISTORICAL NOTE

We have noted earlier (p. 61) the marvelous perfection of Euclid's work and the unparalleled record that, after twenty-two centuries, no system in elementary geometry has been found suitable to supersede his "Elements." It will be interesting to trace the fortunes of this book as it came down to us. We have mentioned the services of Proclus, who wrote a commentary on Euclid in the fifth century A.D., and we are indebted to Theon of Alexandria (378 A.D.) for certain historical information; but these men sought only to explain, not to modify Euclid's Geometry. Their work possesses priceless value as evidence that both the copyists of the Middle Ages and the translators of the Renaissance reproduced this great classic as they found it. When Alexandria was sacked by the Arabs (640 A.D.) Greek learning sought a refuge in the cities on the Mediterranean, where the Arabian scholars from Bagdad learned of the rescued manuscripts of classic learning. Euclid was translated into Arabic in the reign of Al Mamun and carried by the Moors to their schools in Spain. But the Moors jealously guarded their learning, so that not until the twelfth century did Euclid become known to the Christian world. In 1120 A.D., Athelhard, an English monk, while traveling in Spain, found a copy of the "Elements" and translated it into Latin. Other translations followed, and that made by Johannes Campanus (1260) served as copy for the first edition ever printed (Venice, 1482). Original Greek manuscripts discovered later were found to agree with the Arabic copies, and several are now preserved in scientific collections. The first English edition of Euclid was printed in 1570 from a translation made by Sir Henry Billingsley, later Lord Mayor of London. Other editions followed in rapid succession and the study of geometry spread in the higher schools of learning.

Efforts to improve upon Euclid have been made from time to time. Playfair attacked the axioms and Legendre changed the sequence; but in its essentials, Euclid's system still stands unrivalled as a guide for the beginner in geometry. Important contributions to elementary geometry have been made in modern times. Lobachevsky, for example, in his "Geometric Researches on the Theory of Parallels" (1833–1855), Riemann (1854) and other mathematicians have studied the consequences of replacing the parallel postulate (Secs. 126, 127) by others, and have thus developed "Non-Euclidean Geometries." The axioms of geometry have been studied by Hilbert (1900) and others. All of these works are, however, not substitutes for Euclid as an introduction to geometry, but are additions to the body of geometric truth that may be taken up profitably only by the advanced student. Outlines suitable for this purpose have been given by Veblen, Holgate and Woods.*

---

* In "Monographs on Topics of Modern Mathematics relevant to the Elementary Field," edited by J. W. A. Young; New York, Longmans, Green, & Co., 1911.

BOOK IV

MEASUREMENT OF POLYGONS

340. Measurement. To measure the length of a desk or a room we first select a known length, as a foot rule or a yardstick, and lay off this length one or more times on the length to be measured. When we say that the desk is 4 ft. long, we mean that it is four times as long as a foot rule.

341. Unit of Measure. A *unit of measure* is a magnitude of any kind, selected as a standard of comparison for other magnitudes of the same kind.

342. To *measure* any magnitude is to find by trial *how many times it will contain the unit of measure.*

It may happen that the magnitude to be measured will not contain the unit a whole number of times. In such a case, we apply the unit until a magnitude less than the unit remains to be measured. To measure this we select a smaller unit. For example, if the length of a room is to be measured, we may select the yard as unit of measure. If we find that this unit can be applied 14 times with a remainder less than one yard, we select a smaller unit, the foot, for example, and apply it to the remainder. If the remainder does not contain an even number of feet, we find a second remainder, less than 1 foot in length. To this we apply a still smaller unit, say, an inch. If this is not contained a whole number of times, there is a third remainder, to which we apply a still smaller unit, as a tenth of an inch, and so on. Continuing in this way, we either come to a unit that is contained in the remainder which it measures a whole number of times, or we do not. In the former case, we have measured the length exactly; in the latter case, we may continue the process until we have a remainder as small as we please; for example, less than one tenth of an inch, or less than one millionth of an inch. In such a case we say that we have found the length "within one tenth or one millionth of an inch."

343. In plane geometry, we are concerned chiefly with the measurement of two sorts of magnitudes, — lines (straight or curved) and areas.

In the United States, the chief unit of measure for lines is the length of a certain straight bar, carefully preserved in the Bureau of Standards at Washington, and known as the *yard.* For convenience, other units are also used, like the inch, the foot, the rod and the mile. These secondary units are all defined as simple multiples of the yard, integral or fractional, but other units defined independently of the yard might also be used.

344. To measure straight lines the linear units are used. To measure areas, squares are used as units, having as sides any selected one of the linear units, as a square yard, square foot, square inch, square meter and so on.

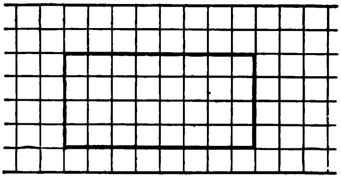
In forty-three countries, including practically all of the leading civilized countries of the world, except the United States and the British Empire, a common system of measure is used in which the chief unit is the length of a straight bar preserved at Paris, and known as the *meter.* The secondary units are the multiples and submultiples of the meter, on the scale of 10. This system is variously known as the *international, decimal* or *metric system.* It is called *international* because so many nations use it in common; *decimal* because it proceeds on the scale of 10, thereby greatly simplifying calculations; *metric*, from meter, the name of the chief unit.

The international system has been made permissible legally in the United States, and is at present used in this country in scientific work, in foreign commerce, and in other work of an international character.

345. To measure curved lines the linear units are also used. But we often need to measure arcs of circles in another system of units, *i.e.* degrees, minutes and seconds. Arcs measured in the latter system serve also to measure angles.

346. Preparatory.

1. Draw a rectangle on squared paper, having its sides coincident with certain lines of the ruling, as in the cut. By counting find how many small squares the rectangle contains.

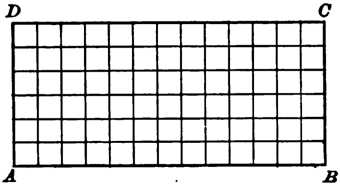


2. Repeat Exercise 1 for several different rectangles.

3. State how to determine the number of squares without counting them all.

PROPOSITION I. THEOREM

347. *The number of square units in the area of a rectangle is equal to the product of the number of linear units in its length and the number of like linear units in its breadth.*



Given the rectangle $ABCD$, of which the side AB is l units long, and the side AD is m units long.

To prove that the area of the rectangle is ml square units.

Proof. 1. Divide the sides AB and AD into l and m equal divisions respectively, and draw straight lines between the corresponding points of division on opposite sides. Then, the smallest figures so formed are all equal squares. Sec. 162

2. In each of the rows parallel to the side AB, there are l such squares, and there are m such rows.

3. $\therefore$ there are m times l, or ml such squares in the area of the rectangle. Q.E.D.

Note. — For convenience, this theorem is often stated in the abridged form: *The area of a rectangle is the product of its length and its breadth.* Other similar abridged statements are used for convenience in what follows. In each instance the full form should be given as an exercise.

348. In Sec. 347, the given rectangle may be such that the unit selected, say an inch, is not contained an integral number of times in one or both of the sides. In this case, there will be left one or two strips, as in Fig. 1, each less than an inch wide, whose area will not be included if we count only the complete squares, as above.

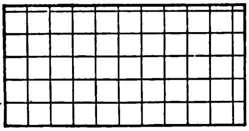


Fig. 1

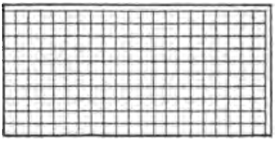


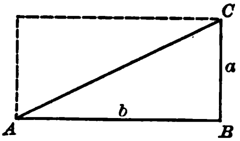
Fig. 2

If we use a smaller unit, say $\frac{1}{2}$ in., this will either (*a*) be contained a whole number of times in both sides and the entire rectangle is divided up into squares, or (*b*) one or both of the sides do not contain the unit exactly; in this case, there are left one or two strips, as in Fig. 2, each less than $\frac{1}{2}$ in. wide, not covered by complete squares.

Selecting a smaller unit, say $\frac{1}{10}$ in. or $\frac{1}{100}$ in., we either measure the area exactly, or have one or two strips left, each less than $\frac{1}{10}$ in. or $\frac{1}{100}$ in. wide as the case may be. Continuing in this way, we either measure the area exactly or with as close an approximation as we please. It can be proved that the proposition of Sec. 347, suitably interpreted, applies with theoretic accuracy in all cases, and we shall hereafter use the proposition as applying to any rectangle whatever.

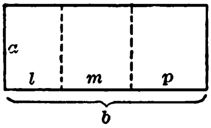
349. Cor. 1. *The area of a right triangle is one half the product of the sides including the right angle.*

For, a diagonal of a rectangle divides it into two congruent right triangles.



350. Cor. 2. *The rectangle formed by two lines is equal to the sum of the rectangles formed by one of the lines and the several segments into which the other is divided.*

Thus, the rectangle ab equals the sum of the rectangles al, am and ap.



351. Cor. 3. *Prove geometrically that :*

$$(x+y)^2 = x^2 + 2xy + y^2.$$

Suggestion. Show from Fig. 1 that the rectangle of sides $x+y$ and $x+y$, or the square of side $x+y$, is the square on x plus the square on y plus twice the rectangle of x and y.

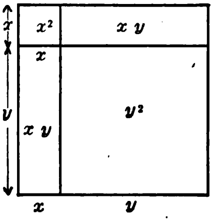


FIG. 1

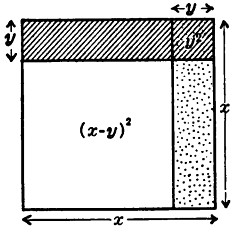


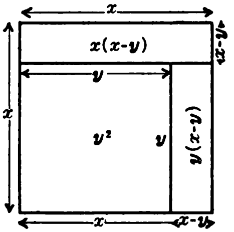
FIG. 2

352. Cor. 4. *Prove geometrically that :*

$$(x-y)^2 = x^2 - 2xy + y^2.$$

Suggestion. In Fig. 2 note that the whole figure is x^2; the entire

part shaded with lines is xy; the entire part dotted is xy; while the small square both shaded and dotted is y^2.



353. Cor. 5. *Prove geometrically that:*

$$(x-y)(x+y) = x^2 - y^2.$$

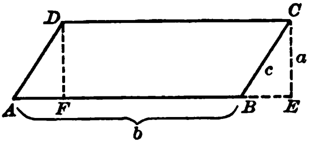
Suggestion. According to the figure the whole area x^2 when diminished by y^2 leaves what two rectangles?

If these rectangles are placed end to end, a rectangle is formed whose width is $x-y$, and whose length is $x+y$. What does the rectangle $(x-y)(x+y)$ equal?

354. In finding areas of parallelograms and triangles, it is necessary to speak of bases and altitudes, and since such figures may have more than one altitude, the altitude belonging or drawn to a given base is designated as the *corresponding altitude.*

PROPOSITION II. THEOREM

355. *The area of a parallelogram is the product of any base and the corresponding altitude.*



Given a parallelogram $ABCD$, and CE the altitude on AB. Let $AB = b$, and $CE = a$.

To prove that area $ABCD = ab$ square units.

Proof. 1. $DF \perp AB$, $CE \perp AB$. $\therefore DF \parallel CE$. Why?

2. $\triangle ADF$ is congruent to $\triangle BCE$. Why?

3. $\therefore$ ▱ $ABCD$ = ▭ $DFEC = ab$. Q.E.D. Why?

EXERCISES

1. State the proposition of Sec. 355 in full. (See note, p. 217.)

2. Show how to bisect a parallelogram by a line parallel to the base.

3. Prove the proposition of Sec. 355 by taking BC as the base, and as the altitude the perpendicular between AD and BC.

PROPOSITION III. THEOREM

356. *The area of any triangle is one half the product of any base and its corresponding altitude.*

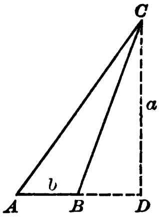


FIG. 1

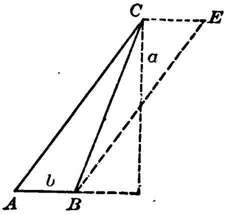


FIG. 2

Given $\triangle ABC$ and let CD be the altitude upon AB. Then, let $CD = a$ and $AB = b$.

To prove that area $ABC = \frac{1}{2} AB \cdot CD$.
$= \frac{1}{2} a \cdot b$.

Proof. 1. Draw $BE \parallel AC$ and $CE \parallel AB$, as in Fig. 2.

2. $ABEC$ is a ▱. Why?
3. $\triangle ABC = \frac{1}{2}$ ▱ $ABEC$. Why?
4. ▱ $ABEC = ab$. Why?
5. $\therefore \triangle ABC = \frac{1}{2} ab$. Why?

357. COR. 1. *If two triangles have equal bases and equal corresponding altitudes, the triangles have the same area.*

358. Cor. 2. *The areas of triangles having equal bases are to each other as the corresponding altitudes of the triangles, and triangles having equal altitudes are to each other as their bases.*

359. Cor. 3. *The areas of two triangles are to each other as the products of their bases and altitudes.*

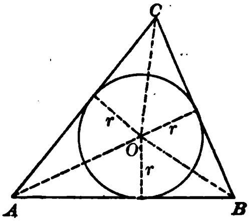
360. Cor. 4. *In a right triangle, the product of the two sides equals the product of the hypotenuse and the altitude upon that side.*

For, the area of the triangle equals one half of either of these products.

361. Cor. 5. *The area of a triangle equals one half the product of its perimeter by the radius of the inscribed circle.*

Express the area of each triangle, OAB, OBC and OCA in terms of r and the respective sides of ABC. Then, add the results.

$$\triangle ABC = \tfrac{1}{2} r(BC + AC + AB).$$



362. Cor. 6. *The area of any polygon circumscribed about a circle is equal to the product of one half of its perimeter by the radius of the inscribed circle.*

363. **Equivalent Figures.** Figures having the same area are said to be **equivalent.**

EXERCISES

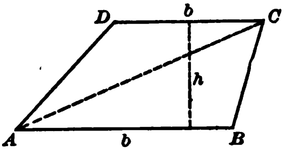
1. Any median of a triangle divides it into two equivalent triangles.

2. The diagonals of a parallelogram divide it into two pairs of equivalent triangles.

3. State the proposition of Sec. 356 in full. (See note, page 217.)

PROPOSITION IV. THEOREM

364. *The area of a trapezoid is half the product of its altitude and the sum of its parallel sides.*



Given trapezoid $ABCD$, and the bases $AB = b$ and $CD = b'$.

To prove that $ABCD = \frac{1}{2}h(b + b')$.

Proof. 1. Draw the diagonal AC and altitude h.

2. Then, $\triangle ABC = \frac{1}{2}hb$, Why?

and $\triangle ACD = \frac{1}{2}hb'$. Why?

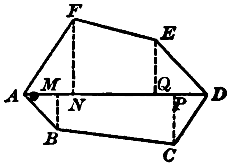
3. $\therefore$ trapezoid $ABCD = \triangle ACD + \triangle ABC = \frac{1}{2}h(b + b')$.

Q.E.D.

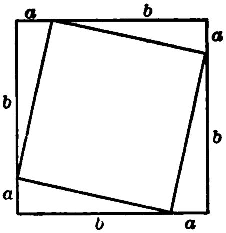
365. COROLLARY. *The area of a trapezoid is equal to the product of its altitude by the line joining the mid-points of the nonparallel sides.*

According to Sec. 166, the line joining the mid-points of the nonparallel sides of a trapezoid equals half the sum of the bases. Hence, the length of this line may be substituted for $\frac{1}{2}(b + b')$ in step (3) of the theorem.

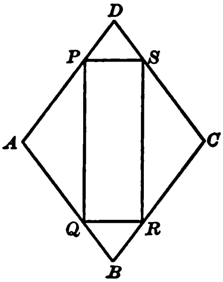
366. Areas of Irregular Polygons. The area of an irregular polygon may be found by dividing it into triangles and by measuring the necessary sides, diagonals and altitudes, to compute the areas of all the triangles. Or, it may be divided into triangles and trapezoids, as in the figure.



EXERCISES



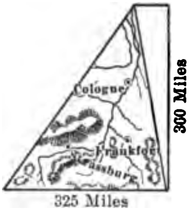
1. From the four vertices of a square, the same distance is laid off on each side, in the same direction around the square. If the four points thus marked are joined in order, prove that the figure formed is a square, and that the four triangles cut off are equal.



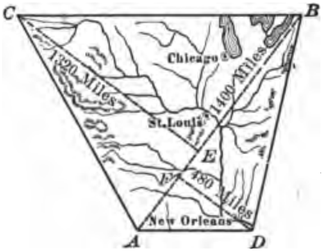
2. From two opposite vertices of a rhombus equal distances are laid off on each side. Prove that a rectangle is formed by joining in order the points thus marked. In the figure

$$PD = DS = BR = BQ.$$

3. In the figure of Exercise 2 show that the diagonals of the rectangle and of the rhombus are concurrent.



4. The approximate area drained by the river Rhine and its tributaries is represented by this triangle; from the dimensions given in the figure find this area in square miles.



5. The approximate area drained by the Mississippi and its tributaries is represented by this quadrilateral; from the dimensions given find the area.

6. The area drained by the river Danube and its tributaries may be represented by two trapezoids, one having bases 200 mi. and 500 mi. and altitude 300 mi., the other having bases 200 mi. and 500 mi. and altitude 600 mi. Find the area of this territory.

7. The base of a right triangle is three times its altitude. The area of the triangle is 96. Find the base and the altitude.

Suggestion. Let h represent the altitude. Then $3h$ represents the base. What represents the area?

8. The perimeter of a certain rectangle is 26 and its area is 40. Find its sides.

Suggestion. Letting a and b represent the sides, we have given that $2a + 2b = 26$, and $ab = 40$. Solving these equations, we obtain a and b.

9. In a right triangle, the sum of the two sides including the right angle is 17, and the area is 35. Find these sides.

10. The base of a right triangle exceeds its altitude by 7. The area of the triangle is 60. Find its base and altitude.

11. In a certain trapezoid the upper base is 5 less than the altitude, and the lower base is 5 greater than the altitude. The area is 64. Find the altitude and the bases.

12. In a certain trapezoid, the lower base exceeds the upper base by 4, and the altitude equals the upper base. The area of the trapezoid is 35. Find its bases and altitude.

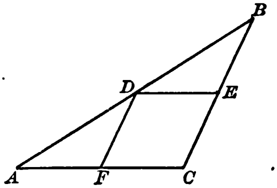
13. In the figure, D is the mid-point of AB, $DE \parallel AC$, $DF \parallel BC$.

Show that: (1) $\triangle ADF = \triangle DEB$.

(2) E is mid-point of BC, and F is mid-point of AC.

(3) Area $ADF = \frac{1}{2}$ area $DFCE$.

(4) If the mid-points of two sides of any triangle are joined, the area of the triangle thus formed is $\frac{1}{4}$ that of the original triangle.



14. Prove that the four triangles into which any parallelogram is divided by its diagonals are equivalent.

15. If the mid-points of two consecutive sides of a parallelogram be joined, the area of the triangle thus formed is $\frac{1}{8}$ of that of the parallelogram.

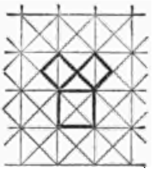
16. If the mid-points of a parallelogram be joined in order, the quadrilateral thus formed is a parallelogram, and its area is $\frac{1}{2}$ of that of the given parallelogram.

17. The mid-points of the sides of a triangle of area 40 are joined in order. What is the area of the triangle formed?

367. Preparatory.

1. What is the size of the largest angle in each little triangle in the figure? What kind of triangles are they? How do they compare with one another in size?

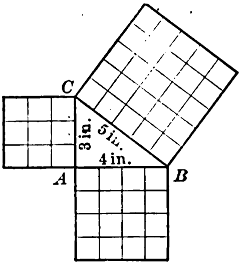
2. If one triangle is singled out, there is a square on each side, as shown by the heavy lines in the figure. How many triangles are there in one of the small squares that border the triangle? In both the small squares? In the large square?



3. Compare the sum of the areas of the small squares with the area of the large square.

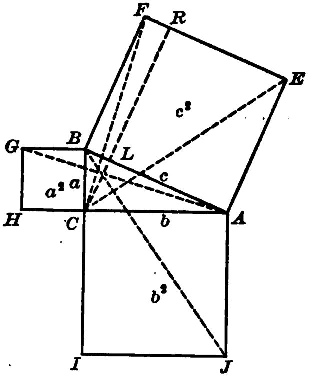
4. In the figure at the right compare the area of the square on the hypotenuse with the sum of the areas of the squares on the other two sides

5. Construct a right-angled triangle having a hypotenuse 10 in. and one side 6 in. Measure the third side. Find the area of the square on each side of this triangle. Compare as in the previous exercise.



PROPOSITION V. THEOREM

368. *The square on the hypotenuse of a right triangle is equal to the sum of the squares on the other two sides.*



Given. The triangle ABC, with right angle at C. Let a, b, c, denote the lengths of the sides as indicated, measured in the same unit.

To prove that $a^2 + b^2 = c^2$.

1. Draw $CR \perp FE$ and connect C with F, C with E, B with J, and A with G.

2. In $\triangle\text{s}$ ABG and FBC, $\angle GBA = \angle CBF$, each being $90° + \angle CBA$; side $BG = CB$, and $BF = BA$. Hence $\triangle ABG$ is congruent to $\triangle FBC$.

3. Area $ABG = \frac{1}{2}$ area $BGHC$.
For they have the same base and altitude

4. Area $FBC = \frac{1}{2}\, FBLR$. Similarly

5. $\therefore$ area $BGHC =$ area $FBLR$ or $a^2 =$ area $FBLR$.
By step (2)

6. $\triangle ECA$ is congruent to $\triangle ABJ$. As in step 2

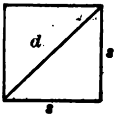
7. Area $AJCI =$ area $ALRE$ or $b^2 =$ area $ALRE$.
As in steps (3), (4), (5)

8. But area $FBLR$ + area $ALRE$ = area $ABFE$.

9. Hence, $a^2 + b^2 = c^2$. Steps (5) and (7)

369. Cor. 1. *The square on either side of a right triangle is equivalent to the square on the hypotenuse diminished by the square on the other side.*

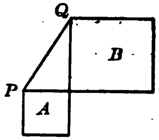
370. Cor. 2. *The diagonal of any square is equal to $\sqrt{2}$ times the side.*



For, $d^2 = s^2 + s^2 = 2\,s^2$,

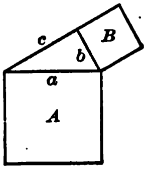
or, $d = s\sqrt{2}$.

371. Cor. 3. *To construct a square equivalent to the sum of two given squares.*



Place the squares like A and B in the figure and construct a square on PQ.

372. Cor. 4. *To construct a square equivalent to the difference between two given squares.*



On side a of square A, construct a right triangle with one side equal to b. On c construct the required square.

373. Cor. 5. *To construct a square equivalent to the sum of n given squares.*

Apply Corollary 3 repeatedly.

EXERCISES

Compute the unknown sides of the following right triangles. The square roots may all be found by factoring. The relation $a^2 - b^2 = (a + b)(a - b)$ will shorten calculation.

	1.	**2.**	**3.**	**4.**	**5.**	**6.**	**7.**	**8.**	**9.**	**10.**
$c =$	5	13	x	41	113	x	145	315	89	235
$a =$	4	x	6	9	x	44	x	252	39	z
$b =$	x	12	8	x	15	117	17	x	x	141

11. In Fig. 1, the outer figure is a square of side $a+b$. a and b are sides of triangle T. Figure 2 is the same square. Show that it equals a^2+b^2+4T. From these two squares, show that $a^2+b^2=c^2$.

Suggestion. Show that the inner quadrilateral in Fig. 1 is the square on the hypotenuse c, and that the whole square equals c^2+4T.

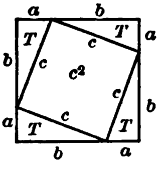


FIG. 1.

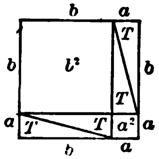
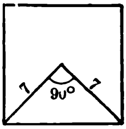
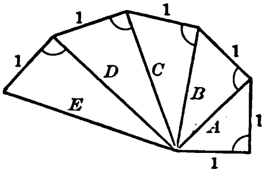


FIG. 2.



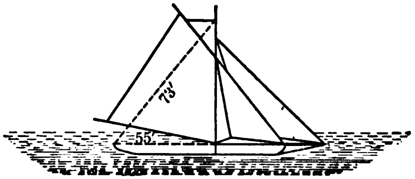
12. Find the area of the square. It is not necessary to extract square root.

13. The triangles in this figure are right triangles. Find the lengths of A, B, C, D, E.



14. A rope 73 ft. long is fastened to the top of a boat's mast, and when drawn taut, it touches a point on the deck 55 ft. from the foot of the mast. Find the height of the mast.

15. A garden has the form of a right triangle. Its hypotenuse is 149 ft. and one side 51 ft. What will it cost to surround it with a fence at 70¢ a running foot?



16. If a ladder 41 ft. long rests against a wall with its foot 9 ft. from the wall, how high is the top of the ladder from the ground?

17. Construct a square whose area is five times that of a given square.

Suggestion. Construct a right triangle with a base equal to the side of the given square and an altitude twice that side. Construct a square on the hypotenuse of this triangle and apply Sec. 368.

Find the lengths of the diagonals, to two decimal places, of squares whose sides are:

18. 15 in. **19.** $2\frac{1}{4}$ ft. **20.** $\frac{3}{4}$ yd.

Find similarly the sides of squares whose diagonals are:

21. 10 ft. **22.** $11\frac{1}{2}$ in. **23.** $6\frac{1}{8}$ ft.

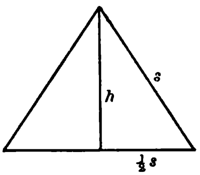
Given the hypotenuse and one side of a right triangle, find the other side, to two decimal places, in each case:

24. $c = 36$ in.
$a = 21$ in.

25. $c = 48$ ft.
$b = 24$ ft.

26. $a = 10$ yd.
$c = 80$ yd.

27. Find the altitude h of an equilateral triangle expressed in terms of its side s.



28. Find the side s of an equilateral triangle of altitude a.

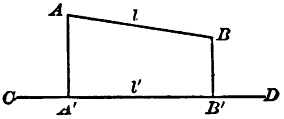
29. Find the area of the equilateral triangle of Exercise 27.

30. Find the length of the shortest chord that can be drawn through a point 4 in. from the center of a circle whose radius is 5 in. Find the length of the longest chord.

31. Find, to two decimal places, the lengths of the chords from either end of the shorter chord of Exercise 30 to the ends of the longer chord.

32. If the diagonals of a quadrilateral are perpendicular to each other, the sum of the squares on one pair of opposite sides equals the sum of the squares on the other pair.

374. Projections. If from the ends of a line perpendiculars are drawn to another line, the segment thus intercepted is called the **projection** of the first line upon the second.



Thus, $A'B'$ or l' is the projection of AB or l upon the line CD. The point A' is the projection of A, and B' is the projection of B. If a point traces the line l by moving from A to B, its projection will trace the line l' by moving from A' to B'.

PROPOSITION VI. THEOREM

375. *In any triangle the square on the side opposite to an acute angle is equivalent to the sum of the squares on the other two sides diminished by twice the rectangle of either of those sides and the projection of the other upon it.*

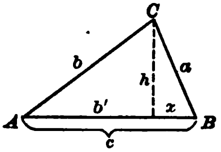


FIG. 1

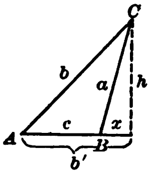


FIG. 2

Given triangle ABC with an acute angle at A. Let b' be the projection of b upon c.

To prove that $a^2 = b^2 + c^2 - 2\,b'c$.

Proof. 1. $x = c - b'$. Fig. 1 by hypothesis
$x = b' - c$. Fig. 2 by hypothesis

2. $x^2 = b'^2 + c^2 - 2\,b'c$, forming the square in step (1). Sec. 352

3. $x^2 = a^2 - h^2$, and $h^2 = b^2 - b'^2$. Sec. 369

4. $\therefore x^2 = a^2 - b^2 + b'^2$. Step (3).

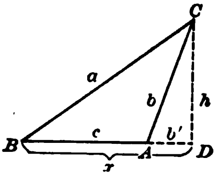
5. $\therefore a^2 - b^2 + b'^2 = b'^2 + c^2 - 2\,b'c$. Steps (2) and (4).

6. $\therefore a^2 = b^2 + c^2 - 2\,b'c$. Step (5). Q.E.D.

EXERCISE

Project c upon b, and prove the theorem.

376. COROLLARY. *In any triangle with an obtuse angle, the square on the side opposite the obtuse angle is equivalent to* *the sum of the squares on the other two sides increased by twice the rectangle of either of these sides and the projection of the other upon that side.*



Proof. 1. $x = c + b'$.

2. $x^2 = c^2 + b'^2 + 2\,b'c$. Squaring (1). Sec. 351

3. But $x^2 = a^2 - b^2 + b'^2$, from step (4) of the theorem.

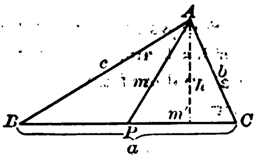
4. $\therefore a^2 = b^2 + c^2 + 2\,b'c$. Steps (2) and (3). Q.E.D.

EXERCISES

1. In Proposition VI suppose $\angle B$ is a right angle. What is the relation of b' to c? Substitute the value of b' in step (6) of the proof of Proposition VI. Show that the result is Proposition V.

2. The difference between the squares on any two sides of a triangle is equivalent to the difference between the squares of the projections of these sides upon the third side of the triangle.

3. The sum of the squares on two sides of a triangle is equivalent to twice the square on half the third side, increased by twice the square on the median to that side.



Suggestion. In the triangle ABC let $c > b$, and m' the projection of m upon a.

To prove that

$$c^2 + b^2 = 2\,\overline{BP}^2 + 2\,m^2.$$

$\angle APB$ is obtuse and $\angle CPA$ is acute. Why?

m lies between c and h, since $c > b$.

$$c^2 = \overline{BP}^2 + m^2 + 2\,\overline{BP} \cdot m'. \qquad \text{Sec. 376}$$

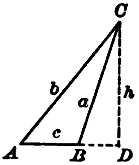
$$b^2 = \overline{PC}^2 + m^2 - 2\,\overline{PC} \cdot m'. \qquad \text{Sec. 375}$$

Add these equals and note that $BP = PC$.

4. The difference between the square on two sides of a triangle is equivalent to twice the rectangle of the third side and the projection of the median to that side.

Suggestion. Subtract the last equality from the one above it in the suggestion for Exercise 3.

5. Express the area of a triangle in terms of its sides.



Solution. Let A be the acute angle. Then:

1. In $\triangle ADC$, $h^2 = b^2 - \overline{AD}^2$. Sec. 369

2. In $\triangle ABC$, $a^2 = b^2 + c^2 - 2 c \cdot AD$. Sec. 375

3. $\therefore AD = \dfrac{b^2 + c^2 - a^2}{2c}$.

4. $$\therefore h^2 = b^2 - \frac{(b^2 + c^2 - a^2)^2}{4c^2} = \frac{4b^2c^2 - (b^2 + c^2 - a^2)^2}{4c^2}$$

$$= \frac{(2bc + b^2 + c^2 - a^2)(2bc - b^2 - c^2 + a^2)}{4c^2}$$ Steps (3) and (1)

$$= \frac{\{(b + c)^2 - a^2\}\{a^2 - (b - c)^2\}}{4c^2}$$

$$= \frac{(a + b + c)(b + c - a)(a + b - c)(a - b + c)}{4c^2}.$$

5. To make the value of h more compact, let $a + b + c = 2s$, where s is the semiperimeter of the triangle.

Then, $b + c - a = 2s - 2a = 2(s - a)$. (1)

Find $a + b - c =$ (2)

Find $a + c - b =$ (3)

and substitute (1), (2) and (3) in step 4.

Then, $$h = \frac{2}{c}\sqrt{s(s - a)(s - b)(s - c)}.$$

6. Place this value of h in the formula for the area $\frac{1}{2} hc$.

It follows from the value of h in Ex. 5 and by taking the squares of the sides b and c, according to Proposition VI and corollary, that all of the altitudes of a triangle may be expressed in terms of the sides of a triangle.

6. Use the formula of Ex. 5 to find the area of a triangle whose sides are 6, 8 and 10.

7. Solve the equation of Exercise 3 for m and express it in terms of a, b, c.

8. Using the formula of Exercise 7 compute the medians of the triangle, to two decimal places, whose sides are 3 in., 4 in., 5 in.

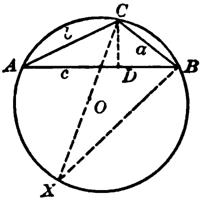
9. Express the radius of the circle circumscribed about a triangle in terms of the sides of the triangle.

Suggestion. Draw the diameter $CX = 2R$ and connect X with B. Prove $\triangle ADC$ similar to $\triangle BXC$. Then,

$$CA : CX = CD : CB,$$

or, $$CA \cdot CB = CD \cdot CX,$$

or, $$ab = CD \cdot 2R.$$



Substitute the value of CD, or h, in terms of a, b, c, as found in Exercise 5, and solve the equation for R. Express the result in this form,

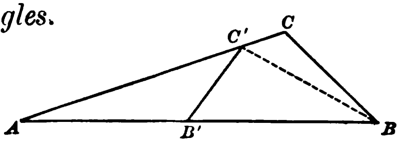
$$R = \frac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}}.$$

Using h in Ex. 9, compute the radii, to two decimal places, of the circles circumscribed about the triangles of sides:

10. 6 in., 8 in., 10 in. **11.** 9 ft., 12 ft., 15 ft.

PROPOSITION VII. THEOREM

377. *The ratio of the areas of two triangles that have an angle of the one equal to an angle of the other, equals the ratio of the products of the sides including the equal angles.*



Given the $\triangle ABC$ and $AB'C'$ with the common angle A.

To prove that $\dfrac{\triangle ABC}{\triangle AB'C'} = \dfrac{AB \cdot AC}{AB' \cdot AC'}$.

Proof. 1. With $\angle A$ on $\angle A'$, connect C' and B.

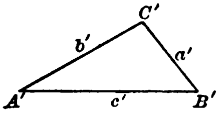
2. $\dfrac{\triangle ABC'}{\triangle AB'C'} = \dfrac{AB}{AB'}$. Sec. 358

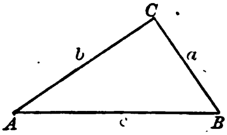
3. $\dfrac{\triangle ABC}{\triangle ABC'} = \dfrac{AC}{AC'}$. Sec. 358

4. $\therefore \dfrac{\triangle ABC}{\triangle AB'C'} = \dfrac{AB \cdot AC}{AB' \cdot AC'}$, by taking the product of steps (2) and (3). Q.E.D.

PROPOSITION VIII. THEOREM

378. *The ratio of the areas of two similar triangles equals the ratio of the squares of any two corresponding sides.*





Given two similar triangles ABC and $A'B'C'$.

To prove that $\frac{\triangle ABC}{\triangle A'B'C'} = \frac{a^2}{a'^2}$ or $\frac{b^2}{b'^2}$ or $\frac{c^2}{c'^2}$.

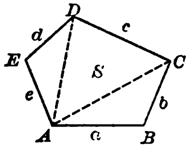
Proof. 1. $\frac{\triangle ABC}{\triangle A'B'C'} = \frac{ab}{a'b'}$. Sec. 377

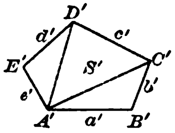
2. $\frac{ab}{a'b'} = \frac{a \cdot a}{a' \cdot a'} = \frac{a^2}{a'^2}$. Sec. 308

3. $\therefore \frac{\triangle ABC}{\triangle A'B'C'} = \frac{a^2}{a'^2} = \frac{b^2}{b'^2} = \frac{c^2}{c'^2}$. Steps (1) and (2). Q.E.D.

PROPOSITION IX. THEOREM

379. *The ratio of the areas of two similar polygons equals the ratio of the squares of any two corresponding sides.*





Given two similar polygons $ABCDE$ and $A'B'C'D'E'$. Let their areas be S and S', and their sides a, b, c, d, e, and a', b', c', d', e'.

To prove that $\frac{S}{S'} = \frac{a^2}{a'^2} = \frac{b^2}{b'^2}$, and so on.

Proof. 1. $\frac{\triangle ABC}{\triangle A'B'C'} = \frac{a^2}{a'^2} = \frac{b^2}{b'^2} = \frac{ac}{a'c'}$. Secs. 320, 378

2. Similarly $\frac{\triangle ACD}{\triangle A'C'D'} = \frac{\overline{AD}^2}{\overline{A'D'}^2} = \frac{c^2}{c'^2} = \frac{ac}{a'c'}$.

3. And $\frac{\triangle ADE}{\triangle A'D'E'} = \frac{\overline{AD}^2}{\overline{A'D'}^2} = \frac{d^2}{d'^2} = \frac{e^2}{e'^2} = \frac{ac}{a'c'}$.

4. $\therefore \frac{\triangle ABC}{\triangle A'B'C'} = \frac{\triangle ACD}{\triangle A'C'D'} = \frac{\triangle ADE}{\triangle A'D'E'} = \frac{a^2}{a'^2}$.

5. But $\frac{\triangle ABC + \triangle ACD + \triangle ADE}{\triangle A'B'C' + \triangle A'C'D' + \triangle A'D'E'} = \frac{a^2}{a'^2}$. Sec. 290

6. $\therefore \frac{S}{S'} = \frac{a^2}{a'^2}$ or $\frac{b^2}{b'^2}$ or $\frac{c^2}{c'^2}\cdots$. Q.E.D.

380. Cor. 1. *The areas of two similar polygons are to each other as the squares of corresponding diagonals, or other correspondingly placed lines.*

381. Cor. 2. *Two corresponding sides of two similar polygons have the same ratio as the square roots of the areas of the polygons.*

EXERCISES

1. Triangles ABC and DEF are similar. AB and DE are opposite equal angles and $DE = 4\,AB$. The area of ABC is 5 sq. in. Find the area of DEF.

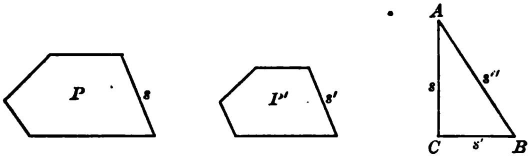
2. Two similar hexagons have areas 49 and 81 respectively. If one side of the first is 14, what is the corresponding side of the other?

3. In triangles ABC and DEF, $\angle A = \angle D$, $AB = 4$, $AC = 7$, $DE = 8$, $DF = 3$. Given that the area of ABC is 12, find the area of DEF.

Suggestion. Apply Sec. 377.

PROPOSITION X. PROBLEM

382. *To construct a polygon similar to two similar polygons and equivalent to their sum.*



Given two similar polygons, P and P', with corresponding sides s and s'.

To construct P'' similar to P and P' and equivalent to $P+P'$.

Construction. 1. Draw a right triangle of sides s and s'.

On AB, or s'', construct a polygon, P'', similar to P. Then P'' is the required polygon. Sec. 339

Proof. 1. $s^2 + s'^2 = s''^2$. Sec. 368

2. $\dfrac{P}{P''} = \dfrac{s^2}{s''^2}$. Sec. 379

3. $\dfrac{P'}{P''} = \dfrac{s'^2}{s''^2}$. Secs. 310, 368

4. $\dfrac{P+P'}{P''} = \dfrac{s^2+s'^2}{s''^2}$, adding (2) and (3).

5. Then, $\dfrac{P+P'}{P''} = \dfrac{s''^2}{s''^2} = 1$. Step (1)

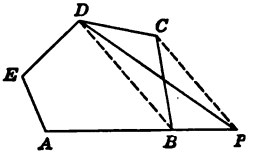
6. $\therefore P + P' = P''$. Q.E.D.

EXERCISE

Explain how to construct a polygon similar to n given similar polygons and equivalent to their sum.

Proposition XI. Problem

383. *To construct a polygon equivalent to a given polygon and having one fewer sides.*



Given the polygon $ABCDE$.

To Construct $APDE = ABCDE$ and having one less side.

Construction. 1. Take three consecutive vertices, as B, C, D, and draw the diagonal DB.

2. From C draw a line $CP \parallel DB$, and meeting AB extended at P. Then, $APDE$ is the required polygon.

Proof. 1. $\triangle DBC = \triangle DBP$. Sec. 357

2. $\therefore APDE = ABCDE$, replacing $\triangle DBC$ by $\triangle DBP$.

Q.E.D.

384. Corollary. *To construct a triangle equivalent to a given polygon.*

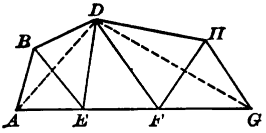
Repeat the process of the theorem until a triangle is formed.

EXERCISES

In the figure, $HG \parallel DF$, and $AB \parallel DE$:

1. Show that quadrilateral $EBDG$ has the same area as pentagon $EBDHF$.

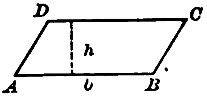
2. Name a triangle that has the same area as quadrilateral $EBDF$.

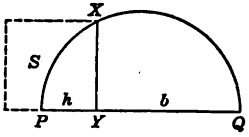


3. Name a triangle that has the same area as quadrilateral $EDHF$.

PROPOSITION XII. PROBLEM

385. *To construct a square equivalent to a given parallelogram.*





Given the parallelogram $ABCD$, with base b and altitude h.

To construct square $\overline{XY}^2 = \square ABCD$.

Construction. 1. Draw a line $PQ = h + b$.

2. Draw a semicircle on PQ and erect $XY \perp PQ$. How?

3. Construct the square S on XY. S is the required square.

Proof. 1. $\overline{XY}^2 = S = hb$. Sec. 324

2. $\therefore S = \square ABCD$. Q.E.D.

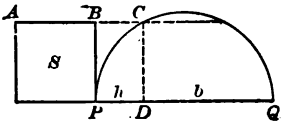
386. COR. 1. *To construct a square equivalent to a given triangle.*

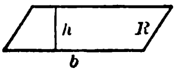
First, construct a parallelogram equal to the given triangle by taking for its base the base of the triangle, and for its altitude one half of the altitude of the triangle. Then, apply the theorem.

387. COR. 2. *To construct a square equivalent to a given polygon.*

First, apply to the polygon Sec. 384; then, Sec. 386.

388. COR. 3. *To construct a parallelogram equivalent to a given square and having the sum of its base and altitude a given line.*





Given square S of side AB and the line PQ.

To construct a parallelogram, R equal to S, and having the sum of its altitude and base equal to PQ.

Construction. 1. Extend a side of S, and lay off on it PQ.

2. On PQ draw a semicircle.

3. Extend side AB of S to cut the circle in C.

4. Draw $CD \perp PQ$.

5. Draw a ▱ having DQ as the base and PD as the altitude.

R is the required parallelogram.

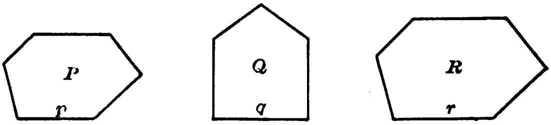
Proof. 1. $\overline{CD}^2 = hb$. Sec. 324

2. But, $\overline{CD}^2 = S$. Construction Steps (1) and (4).

3. $\therefore S = hb$. Q.E.D.

PROPOSITION XIII. PROBLEM

389. *To construct a polygon similar to a given polygon and equivalent to another given polygon.*



Given polygons P and Q, with corresponding sides, p and q.

To construct a polygon R similar to P and equivalent to Q.

Construction. 1. Construct squares equivalent to P and Q, and denote their sides by s and s'. Sec. 387

2. Construct the fourth proportional to s, s' and p and call it r. Sec. 302

3. On r construct a polygon R similar to P. Sec. 339

R is the required polygon.

Proof. 1. From step (2) of the construction

$$s : s' = p : r.$$

Hence, $s^2 : s'^2 = p^2 : r^2$. Sec. 289, IV

2. Then, area P : area $Q = p^2 : r^2$. Step (1) of construction.

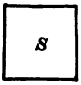
But, area P : area $R = p^2 : r^2$. Sec. 379

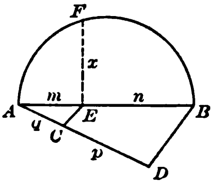
3. $\therefore$ area P : area Q = area P : area R, Step (2)

or, area Q = area R. Why? Q.E.D.

PROPOSITION XIV. PROBLEM

390. *To construct a square that shall have a given ratio to a given square.*





Given the square S and the ratio $\frac{p}{q}$.

To construct a square S', such that area S' : area $S = p : q$.

Construction. 1. Draw m equal to a side of square S and draw AD at a convenient angle to m.

2. On AD lay off q and p. Connect C with E and draw $DB \parallel CE$. Extend m to meet DB.

3. On AB draw a semicircle and draw $EF \perp AB$.

EF is the side of the required square.

Proof. 1. $m : x = x : n.$ Sec. 324

2. $m^2 : x^2 = x^2 : n^2 = mn : n^2 = m : n$ from (1)

3. But $q : p = m : n.$ Sec. 294

4. $\therefore$ $m^2 : x^2 = q : p.$ Steps (2) and (3)

5. Or, $x^2 : m^2 = p : q.$ Sec. 289, IX. Q.E.D.

391. COROLLARY. *To construct a polygon similar to a given polygon and having a given ratio to it.*

Let P be the given polygon of side s, and let the ratio be $\frac{p}{q}$.

Draw a line s' such that $s'^2 : s^2 = p : q$. Sec. 390

On s' construct a polygon R similar to P. Sec. 339

R is the required polygon.

For, area R : area $P = s'^2 : s^2$. Sec. 379

But, $s'^2 : s^2 = p : q$. Construction

$\therefore$ area R : area $P = p : q$. Q.E.D.

EXERCISES

Compute the areas of the following triangles:

	1.	2.	3.	4.	5.
Altitude =	12 ft.	15.5 in.	6 8 in.	1 ft. 4 in.	17½ yd.
Base =	26 ft.	12.4 in.	4 ft.	2 ft. 8 in.	5⅓ yd.

Compute the areas of the following parallelograms:

	6.	7.	8.	9.	10.
Altitude =	13 in.	17.4 ft.	6½ ft.	11 in.	3⅝ ft.
Base =	25 in.	15.3 ft.	1.25 ft.	3 ft. 8 in.	4⅙ yd.

Compute the square on the hypotenuse, to two decimal places, if necessary, of each of the following right triangles:

	11.	12.	13.	14.	15.
Side a =	17 in.	6 ft.	3 ft. 6 in.	10 in.	12.5 yd.
Side b =	20 in.	8 ft.	6 ft. 2 in.	1 ft. 8 in.	9.5 yd.

Compute the areas of the following trapezoids:

	16.	17.	18.	19.	20.
Upper base =	12 in.	7 ft.	18 in.	3 ft. 8 in.	7½ yd.
Lower base =	16 in.	30 ft.	2½ ft.	1 ft. 10 in.	3⅓ yd.
Altitude =	4.5 in.	40⅓ ft.	42 in.	6 ft. 6 in.	12¾ yd.

21. The areas of two similar polygons are respectively 18 sq. in. and 1 sq. ft. Find the ratio of the corresponding sides.

22. The areas of two triangles of equal bases are respectively 20.5 sq. ft. and 5 sq. ft. What is the ratio of their altitudes?

23. Three similar polygons are drawn on the three sides of a right triangle. The areas of those on the perpendicular sides are respectively 4 sq. ft. 48 sq. in. and 10 sq. ft. 96 sq. in. Find the area of the polygon on the hypotenuse.

24. The three sides of a triangle are 8.1 in., 7.9 in., 8.3 in. Is the largest angle acute, right or obtuse?

25. Find the area, to two decimal places, of an isosceles triangle whose base is 2 ft. and whose perimeter is 10 ft.

26. Find the area of a right triangle whose hypotenuse is 15 in. and one of whose perpendicular sides is 9 in.

27. Find the area, to two decimal places, of an equilateral triangle whose perimeter is 24 in.

28. Find the area of an isosceles right triangle whose hypotenuse is $3\sqrt{2}$ ft.

29. Find the area of an equilateral triangle each of whose sides is $2\sqrt{3}$ yd.

30. The equilateral triangle constructed upon the hypotenuse of a right triangle is equivalent to the sum of the equilateral triangles constructed upon the other sides of the triangle.

31. The area of a rhombus is one half the product of its diagonals.

32. From a point within a parallelogram lines are drawn to the four vertices, the sum of either pair of triangles with parallel bases is equivalent to the other pair.

33. The line joining the mid-points of the parallel sides of a trapezoid bisect the trapezoid.

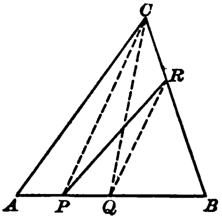
34. On a given line construct a triangle equivalent to a given triangle.

35. On a given line as hypotenuse construct a right triangle equivalent to a given triangle.

36. Construct a right triangle equivalent to any given triangle.

37. Draw a line through a given point in one of the sides of a triangle so as to bisect the area of the triangle.

Suggestion. Let the point be P on side AB of triangle ABC. The median CQ bisects the triangle. Connect P with C and draw through Q a line parallel to CP. Connect R with P and prove that PR is the required line.



38. Construct a square equivalent to half a given square.

39. Divide a triangle into two equivalent parts by a line parallel to the base.

40. If the mid-point of one of the nonparallel sides of a trapezoid is joined to the ends of the other side, the triangle formed is equivalent to one half of the trapezoid.

41. Find the area of an isosceles trapezoid whose bases are b and b' and whose sides each equals s.

42. Divide a parallelogram into two equivalent parts by a line from a point in one of the sides.

43. Transform a pentagon into an equivalent isosceles triangle.

44. If two triangles have an angle of the one supplementary to an angle of the other, the areas of the triangles are proportional to the products of the sides containing the supplementary angles.

45. Two similar triangles are to each other as the squares of two corresponding medians.

46. A polygon whose area is 160 sq. in. has one side 8 in. long. What is the length of the corresponding side of a similar polygon whose area is 40 sq. in.?

47. Construct an equilateral triangle equivalent to two equilateral triangles.

48. Construct a triangle similar to a given triangle and equivalent to a given parallelogram.

49. Construct an equilateral triangle equivalent to a given parallelogram.

50. The sum of the perpendiculars from any point within a convex polygon upon all of its sides is the same for all positions of the point.

Suggestion. Connect the point with all the vertices of the polygon and consider the sum of the areas of all the triangles thus formed.

51. The hypotenuse of a right triangle is 9 ft. and the projection of one of the other sides upon the hypotenuse is 5 ft. Find the area of the triangle.

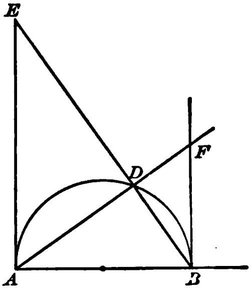
52. Divide a right triangle into two isosceles triangles.

53. The ratio of similitude of two triangles is $\frac{4}{9}$. Find the ratio of their areas.

54. The sides of a triangle are 5, 7, 9. The points where the bisectors of its angles meet the opposite sides are taken as vertices of a new triangle. Find the ratio into which each side of this triangle is divided by the bisectors of the corresponding angles of the given triangle.

55. From the extremities *A*, *B*, of a diameter, lines are drawn to any point *D* of a circle, and produced until they meet the tangents at *A* and *B* in *E* and *F*. If r is the radius of the circle, prove that $BF \cdot AE = 4\,r^2$.

Suggestion. Prove that ▲ *BFD* and *BAE* are similar.

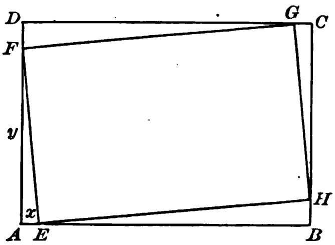


56. Calculate the three altitudes of an isosceles triangle, if each of the equal sides is 13 in. and the base 24 in. in length.

57. In a rectangle whose dimensions are 7 and 10, a rectangle is inscribed whose dimensions are in the ratio of 5 to 8. Find the vertices of the inscribed rectangle.

Solution. Let x and y be the distances of the corners of the inscribed rectangle from the corner *A* of the given rectangle as indicated in the figure.

Then, $\frac{FE}{FG} = \frac{5}{8}$. (1)



Since the angle EFG is 90°, $\angle DFG$ is the complement of $\angle AFE$ and therefore right triangles AEF and DFG are similar and we have:

$$\frac{x}{DF}=\frac{FE}{FG} \qquad (2)$$

or, by (1)

$$\frac{x}{7-y}=\frac{5}{8}. \qquad (3)$$

Likewise,

$$\frac{y}{DG}=\frac{FE}{FG}, \qquad (4)$$

or, by (1),

$$\frac{y}{10-x}=\frac{5}{8}. \qquad (5)$$

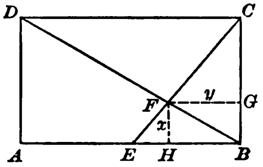
Solve (4) and (5) for x and y.

Then, $x=\frac{10}{13}$, $y=\frac{75}{13}$.

58. Solve the preceding problem if the dimensions of the given rectangle are 20 and 28, and those of the inscribed rectangle are in the ratio of 4 to 7.

59. Solve the same problem if the dimensions of the given rectangle are a and b, and those of the inscribed rectangle are in the ratio of l to m.

60. In the rectangle $ABCD$, $AB=16$ in., $BC=10$ in., E is the midpoint of AB. If F denotes the intersection of BD and CE, find the distance of F from AB and BC.



Solution. From the similar triangles BFG and BAD,

$$\frac{BG}{GF}=\frac{AD}{BA}. \qquad (1)$$

$$\frac{x}{y}=\frac{10}{16}. \qquad (2)$$

From the similar triangles EHF and EBC,

$$\frac{EH}{HF}=\frac{EB}{BC}\cdot \qquad (3)$$

$$\frac{8-y}{x}=\frac{8}{10}. \qquad (4)$$

From (2),

$$x=\frac{5}{8y}. \qquad (5)$$

Solve (4) and (5).

Then, $x=\frac{16}{3}$ and $y=\frac{10}{3}$.

61. Solve the same problem as 60, if $AB = b$, and $BC = a$.

62. An isosceles triangle has base $2a$ and altitude b. Find the radius of the inscribed circle.

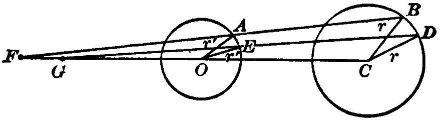
63. In any two non-concentric circles, the straight lines joining the extremities of any two parallel radii lying on the same side of the line of centers, intersect in a common point on the line of centers.

Solution. Let the figure be drawn according to the conditions of the problem, and let G and F be the points where the two lines cut OC. To show that G and F coincide.

We know $$\frac{GO}{GC} = \frac{r'}{r},$$

and $$\frac{FO}{FC} = \frac{r'}{r}.$$

$$\therefore \frac{GO}{GC} = \frac{FO}{FC}. \quad (1)$$



Let $GF = x$.

$\therefore GO = OF - x$.

$GC = FC - x$.

Let $OF = y$, $FC = z$.

$\therefore GO = y - x$.

$GC = z - x$.

$$\therefore \frac{y-x}{z-x} = \frac{y}{z}, \text{ or } yz - xz = yz - xy. \quad (2)$$

$$\therefore xz = xy, \text{ or } x(z-y) = 0.$$

$\therefore$ either $z - y = 0$,

or $x = 0$.

But $z - y$ cannot be 0, since the circles are non-concentric.

$\therefore x = 0$.

64. A point P is situated at distance 42 from the center, A, of a circle of radius 18. Through P, a secant BC is drawn such that the segment, BC, of the secant within the circle equals 13. Find the distance from P to the nearer point of intersection, B.

Suggestion. Connect P and A and produce the line until it cuts the circle again at E. Let PA cut the circle at D between P and A. We have according to the conditions of the problem,

(1) $BC = 13$, $DA = 18$, $PA = 42$.

(2) $PB \cdot PC = PD \cdot PE$. **Sec. 332**

(3) Let $PB = x$.

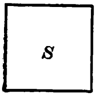
Then (2) becomes by (3) and (1)

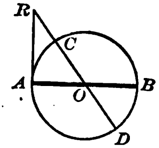
(4) $x(x + 13) = (42 - 18)(42 + 18)$.

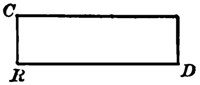
(5) Solve (4).

65. Construct a rectangle equivalent to a given square and having the difference of its base and altitude equal to a given line.

Solution. Let the square be S and line AB.





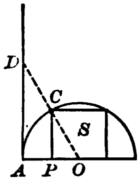


On AB as diameter describe a circle. At A draw a tangent and make $AR =$ a side of S. Draw secant RD through the center of the circle. Then a parallelogram with RD for its base and RC for its altitude is the required rectangle. Prove it.

66. Construct two lines, given their difference and their product.

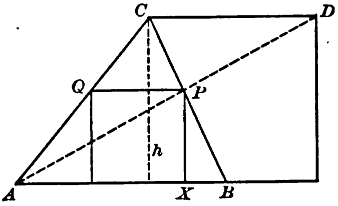
67. Inscribe a square in a semicircle.

Suggestion. Draw the tangent $AD = 2\,AO$ and connect D with O. Prove that CP is the side of the required square.



68. Inscribe a square in a given triangle.

Suggestion. Construct a square on h, and connect D with A. Prove that PX is the side of the required square by showing that $PQ = PX$.



69. Draw a line equal to a given line and having its ends on each of two fixed circles so that its position shall be parallel to the line of centers.

HISTORICAL NOTE.

PYTHAGORAS

Proposition, Sec. 368, is usually called the *Pythagorean Theorem*, but it is by no means the only contribution of Pythagoras to geometry. On the contrary, we are indebted to Pythagoras and his followers for many propositions entirely original with them, whereas the so-called Pythagorean Proposition was known centuries before.

PYTHAGORAS

The Egyptian surveyors, who were compelled to retrace their landmarks after each overflow of the Nile, developed a resourcefulness unique among the early nations, and used the principle of the right triangle in a most effective way. By tying knots in a rope so as to mark lengths in the ratio of 3 to 4 to 5, they erected perpendiculars and outlined their rectangular fields. Pythagoras, however, is given credit for finding the first general proof that the square on the hypotenuse of a right triangle is equivalent to the sum of the squares on the other two sides. The importance attached to this discovery by Pythagoras may be inferred from the tradition that he celebrated it by a special sacrifice of thanksgiving to the gods.

Pythagoras (born about 580 B.C.) was a native of Samos and a pupil of Thales. While yet a young man, he made extensive journeys into foreign countries for the purpose of study, remaining several years in Egypt and probably visiting India. After his return to Samos, and his subsequent failure to secure a following there, he finally settled at Crotona and established the famous Pythagorean school of philosophy. Owing to the vows of secrecy peculiar to this brotherhood, many of the discoveries of Pythagoras's disciples were not disclosed, but it is certain that they constructed the regular pentagon and the pentagram, or five-pointed star, which, surrounded by the letters of the Greek word signifying health, became the badge of their order. Pythagoras knew that six equilateral triangles, or four squares, or three hexagons, fill space about a point and that no other regular polygons, all of the same shape, have this property. This principle was put to practical use in countless designs for tiling and decorating. The Pythagoreans were led to the study of regular figures by their tendency to idealize. They constructed the five regular solids (which are discussed in Solid Geometry) and regarded the circle as the most beautiful of all plane figures.

Political enemies of Pythagoras finally succeeded in overthrowing the brotherhood and further avenged themselves by the murder of the great philosopher (501 B.C.).

BOOK V

REGULAR POLYGONS AND CIRCLES

392. Preparatory. Define:

1. A polygon.
2. A triangle.
3. A quadrilateral.
4. A pentagon.
5. A hexagon.
6. An octagon.
7. A decagon.
8. A dodecagon.

What is the sum of the interior angles of:

9. A triangle?
10. A quadrilateral?
11. An octagon?
12. A polygon of n sides?

393. Equilateral Polygon. A polygon all of whose sides are equal is called an **equilateral polygon.**

394. Equiangular Polygon. A polygon all of whose angles are equal is called an **equiangular polygon.**

395. Regular Polygon. A polygon that is both equilateral and equiangular is called a **regular polygon.**

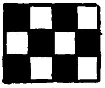
396. An equilateral triangle is necessarily equiangular (Sec. 102) and hence regular. But polygons of more than three sides may be equilateral without being equiangular, and *vice versa.*

EXERCISES

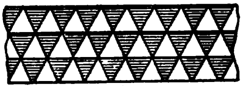
Draw freehand a quadrilateral that is approximately:

1. Neither equilateral nor equiangular.
2. Equilateral but not equiangular.
3. Equiangular but not equilateral.
4. Both equiangular and equilateral; that is, regular.
5. What special name has a regular quadrilateral?

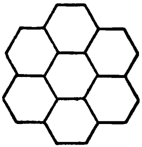
Regular polygons are used in many ornamental designs. Thus:



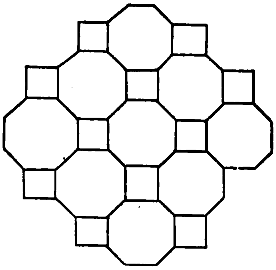
1



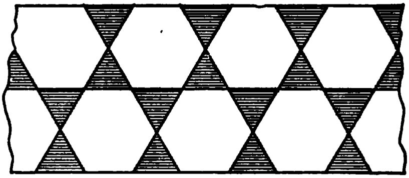
2



3



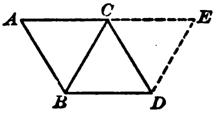
4



5

6. Prove that the above designs can be made up of regular polygons as indicated.

Suggestion. We must show that regular polygons can be fitted together as indicated, without either overlapping or leaving vacant spaces. In design No. 2, for example, we must show that if two equilateral triangles of equal sides be placed as indicated, then *AC* and *BD* are parallel; and if a third triangle is fitted to *CD*, that *ACE* is a straight line, etc. In design No. 4, compute the number of degrees in each angle of a regular octagon and show that octagons and squares may be made to fill space about a point when placed as in the figure. Treat the other designs similarly.



7. What is the perimeter of an equilateral octagon one of whose sides is 4 in. long?

8. The perimeter of an equilateral dodecagon is 84 in. What is the length of each side?

9. The perimeter of an equilateral figure is 60 in. and each side is 6 in. long. How many sides has the figure?

How many degrees in each interior angle of:

10. An equiangular triangle?

11. An equiangular pentagon?

12. An equiangular dodecagon?

13. An equiangular polygon of n sides?

397. Preparatory.

1. How would you divide a given arc and its central angle into two equal parts?

2. How would you divide a given arc and its central angle into four equal parts? Eight equal parts?

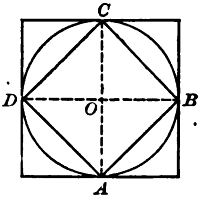
3. If an arc is already divided into a certain number of equal parts, how would you divide it into double that number of equal parts?

4. How would you divide a circle into two equal parts? Four equal parts? Eight? Sixteen? Thirty-two? 2^n equal parts?

398. AXIOM. *A circle may be thought of as divided into any given number,* n, *of equal arcs.*

But the problem of the "Division of the Circle" into n equal parts by actual construction with ruler and compasses can be solved only for certain values of n. These will be pointed out later.

399. Preparatory.



1. If a circle is divided into four equal parts, and the points of division are joined in order, what sort of a figure is formed?

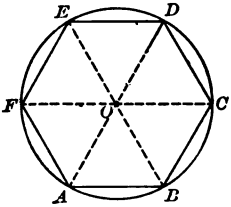
2. In the figure, show that the tangents at the points of division form a square.

PROPOSITION I. THEOREM

400. *If a circle is divided into any number of equal arcs, then,*

(I) *The chords joining the successive points of division form a regular polygon.*

(II) *The tangents at the points of division, produced until they meet, form a regular polygon.*



(I) **Given** arc $AB =$ arc $BC = \cdots =$ arc FA.

To prove that polygon $ABCDEF$ is regular. To do this we must show that the polygon is (1) equilateral; (2) equiangular.

Proof. 1. Chords AB, BC, $\cdots$ are equal since the arcs AB, BC, etc., are equal by hypothesis. Hence, the polygon is equilateral. Sec. 393

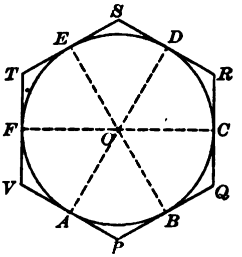
2. Arc $ABC =$ arc BCD. Why?

3. $\therefore \angle ABC = \angle BCD$. Why?

4. Similarly, all the angles of the polygon are equal.

5. Hence, the polygon is regular. Sec. 395. Q.E.D.

(II) **Given** equal arcs *AB*, *BC*, and so on, and *VP* tangent at *A*, *PQ* tangent at *B*, and so on.



To prove that *PQRSTV* is a regular polygon.

Proof. 1. Connect the points *A*, *B*, *C*, ⋯ *F* with the center *O*. Then angles *AOB*, *BOC*, etc., are all equal. Why?

2. Quadrilateral *OAPB* has right angles at *A* and *B*.

$\therefore \angle P = 180° - \angle AOB.$ Secs. 219, 257

3. Similarly, angles *Q*, *R*, *S*, etc., are supplements of the corresponding angles at the center.

4. But central angles *AOB*, *BOC*, etc., are all equal. Step (1)

5. Hence, their supplements are equal, and the polygon *PQRSTV* is equiangular.

6. Place $\angle AOB$ upon $\angle BOC$ so that side *OA* falls on side *OC*, then points *A* and *C* will coincide, and since angles *A*, *B*, *C*, are all 90°, points *P* and *Q* will coincide.

7. $\therefore$ Quadrilaterals *AOBP* and *BOCQ* are congruent, and $PB = BQ$. Similarly, $QC = CR$.

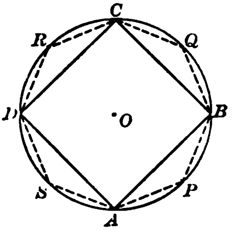
8. But $BQ = QC$. (Why?) Hence, $PQ = QR$.

9. In the same way any two consecutive sides are proved equal. Hence, the polygon is equilateral.

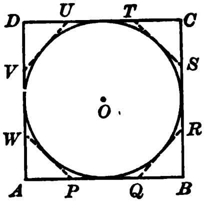
10. $\therefore$ the polygon *PQRSTV* is regular. Sec. 395

401. Cor. 1. *Tangents to a circle at the vertices of a regular inscribed polygon form a regular circumscribed polygon of the same number of sides.*

402. Cor. 2. *If the extremities of each side of a regular inscribed polygon are connected by straight lines with the mid-point of the arc subtended by that side, the polygon so formed is a regular polygon of double the number of sides of the original polygon.*



403. Cor. 3. *If at the mid-points of the arcs joining consecutive points of contact of a regular circumscribed polygon tangents are drawn and produced until they meet the sides of the original polygon, the tangents so drawn together with the portions of the sides of the original polygon between them form a regular polygon of double the number of sides of the original polygon.*



404. Cor. 4. *If at the mid-points of the arcs subtended by the sides of a regular inscribed polygon tangents are drawn, they form a circumscribed regular polygon with sides parallel to the corresponding sides of the inscribed polygon and with vertices on the radii extended through the vertices of the inscribed polygon.*

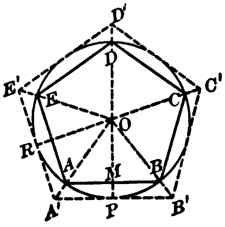
For AB and $A'B'$ are $\perp$ to OM. Why?

$\therefore AB \parallel A'B'$, and similarly for the other sides.

$\triangle ROA'$ is congruent to $\triangle POA'$. Why?

$\therefore$ tangents at R and at P must meet OA at the same point A'. Similarly for the other vertices.

$\therefore A', B', C' \cdots$ lie on the extensions of radii OA, OB, $\cdots$.

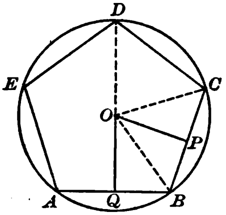


PROPOSITION II. THEOREM

405. *If a polygon is regular:*

(I) *A circle can be circumscribed about it.*

(II) *A circle can be inscribed in it.*



(I) **Given** any regular polygon as $ABCDE$.

To prove that a circle can be circumscribed about it.

It is sufficient to prove that the circle that passes through any three vertices, A, B, C, of the polygon also passes through the other vertices, as D, E.

Proof. 1. Find the center O of the circle through A, B, C.

2. $\angle OBA = \angle OBC = \angle OCB$. Why?

3. $\therefore \angle ABC = \angle BCD$. Why?

4. $\therefore \angle OBA$ or $\angle OCB = \angle OCD$.

5. In $\triangle$s OBC and OCD, OC is common, and $BC = CD$.

6. $\therefore \triangle OBC$ is congruent to $\triangle OCD$. Why?

7. $\therefore OD = OC$, and the circle passes through D.

8. Similarly, the circle passes through all of the vertices.

(II) **Given** the regular polygon $ABCDE$.

To prove that a circle can be inscribed in $ABCDE$.

It is sufficient to find a point that is equidistant from all the sides of the given polygon.

Proof. 1. In the above figure, the sides AB, BC $\cdots$ are equal chords in a circle, therefore they are all equidistant from O, the center of the circumscribed circle. Sec. 212

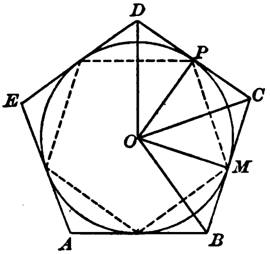
2. $\therefore$ the circle with center O and with OQ, the perpendicular from O to side AB, as the radius will be inscribed in the polygon. Sec. 224. Q.E.D.

406. Cor. 1. *The inscribed and circumscribed circles of a given regular polygon are concentric.*

407. Cor. 2. *The radius drawn to any vertex of a regular inscribed or circumscribed polygon bisects the angle at the vertex.*

408. Cor. 3. *An equilateral polygon inscribed in a circle is a regular polygon.*

409. Cor. 4. *An equiangular polygon circumscribed about a circle is regular.*



For, in the figure, OD bisects $\angle CDE$, OC bisects $\angle BCD$, OB bisects $\angle ABC$, and so on. Sec. 229

$\therefore \angle ODP = \angle OCP = \angle OCM$
$= \angle OBM.$

Also, $CM = CP$. $\therefore$ $\triangle$s CPO and CMO are congruent. Why?

If $\triangle OBC$ is placed on $\triangle OCD$, OM falls on OP and OB on OD, because $\angle OBC = \angle ODP$.
$\therefore BC = CD$.

410. Cor. 5. *The angles at the center subtended by the sides of a regular inscribed or circumscribed polygon are equal and each is supplementary to an angle of the polygon.*

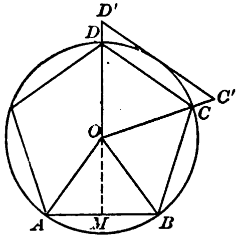
For $\angle AOB = \angle BOC \cdots$. Secs. 208, 203

Also, $\angle BOM = \frac{1}{2} \angle AOB$, and $\angle OBA = \frac{1}{2} \angle ABC$.

But $\angle BOM$ is complementary to $\angle OBA$.

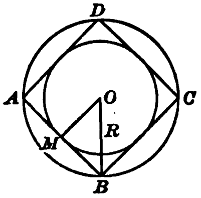
$\therefore \angle AOB$ is supplementary to $\angle ABC$.

The side of the corresponding circumscribed polygon as $D'C'$ subtends the same angle as DC.



411. Center of a Regular Polygon. The common center of the inscribed and the circumscribed circle of a regular polygon is called the **center of the polygon.**

In the figure, O is the center of $ABCD$.



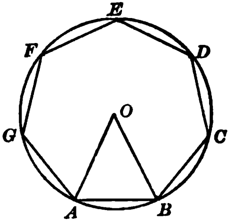
412. Radius of a Regular Polygon. The radius of the circle circumscribed about a regular polygon is called the **radius** of the polygon.

In the figure, $R = OB$ is the radius of $ABCD$.

413. Apothem. The perpendicular from the center of a regular polygon to a side is called the **apothem** of the polygon.

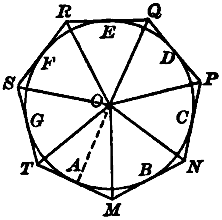
In the figure of Sec. 412 OM is the apothem of $ABCD$.

EXERCISES



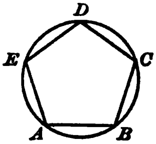
1. Given that arcs AB, BC, etc., are all equal, find the number of degrees in $\angle AOB$; $\angle FGA$; $\angle OAB$.

2. Explain how to inscribe a regular 14-gon in the circle of Exercise 1.



3. If arcs AB, BC, $\cdots$ GA are all equal, find the number of degrees in $\angle NOQ$; $\angle MNP$; $\angle AOM$.

4. Explain how to inscribe a regular 14-gon in the circle of Exercise 3.



5. Show that if a polygon of 5 sides inscribed in a circle is equiangular it is regular.

Suggestion. Since $\angle ABC = \angle BCD, \cdots$, arc ABC = arc $BCD, \cdots$.

$\therefore$ Arc AB = arc CD, arc AB = arc DE, and $BC = DE$

6. Use the same method to show that any polygon of an odd number of sides that is equiangular and that is inscribed in a circle is regular.

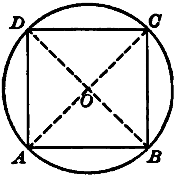
7. Show that not every equiangular polygon of an even number of sides that is inscribed in a circle is regular.

Suggestion. To prove that a statement is not always true, it is sufficient to find one instance in which it is not true.

8. How many degrees in the exterior angle formed by producing a side of a regular polygon of n sides?

PROPOSITION III. PROBLEM

414. *To inscribe a square in a given circle.*



Given the circle with center O.

To inscribe in it a square.

Construction. 1. Draw two diameters at right angles. How? 2. Connect their adjacent ends. $ABCD$ is the required square.

Proof. Apply Sec. 203 and Sec. 205.

415. COR. 1. *To inscribe regular polygons of 8, 16, 32, etc., sides in a circle.*

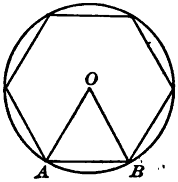
Connect the mid-points of the arcs in the figure, Sec. 414, and repeat this process.

Evidently the number of sides of all regular polygons that can be inscribed according to Proposition III may be expressed by the formula 2^n when n is any integer > 1.

416. Cor. 2. *The side of an inscribed square is equal to the radius of the circle multiplied by* $\sqrt{2}$; *the side of a circumscribed square is equal to twice the radius of the circle.*

Proposition IV. Problem

417. *To inscribe a regular hexagon in a given circle.*



Given the circle with center O.

To inscribe a regular hexagon in the circle.

Construction. 1. At any point A of the circumference describe an arc with a radius equal to OA, cutting the circle again in B.

2. Connect A with B. AB is a side of a regular inscribed hexagon.

3. Repeat the process at B, and so on.

Proof. 1. Connect O with A and B. $\triangle AOB$ is equilateral.

2. $\therefore \angle AOB = 60°$, arc AB is one-sixth of the circumference and AB is one side of a regular inscribed hexagon. Q.E.D.

418. Cor. 1. *To inscribe an equilateral triangle in a given circle.*

Join the alternate vertices of an inscribed regular hexagon.

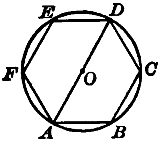
419. Cor. 2. *To inscribe regular polygons of 12, 24, 48, etc., sides in a given circle.*

Connect the mid-points of each arc to the adjacent vertices of a regular inscribed hexagon, and apply the process to the new polygon.

420. Cor. 3. *The side of a regular inscribed triangle is equal to the radius of the circle multiplied by* $\sqrt{3}$; *the side of the regular circumscribed triangle is equal to the radius multiplied by* $2\sqrt{3}$.

421. Cor. 4. *The side of the inscribed regular hexagon is equal to the radius of the circle.*

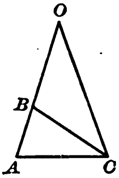
EXERCISES



If *ABCDEF* is equilateral, show:

1. That arc *EDC* = arc *DCB*.
2. That *AOD* is a straight line.
3. That angle *FED* = angle *EDC*.
4. That the hexagon is equiangular.
5. What is the locus of the mid-points of the sides of all regular hexagons inscribed in a given circle?
6. Inscribe a circle in a given regular hexagon.
7. Circumscribe a regular hexagon about a given circle.
8. What is the locus of the vertices of all regular hexagons that can be circumscribed about a given circle?

422. Preparatory.



1. If the vertices of a regular pentagon are joined to the center, how many degrees in each small angle at the center?

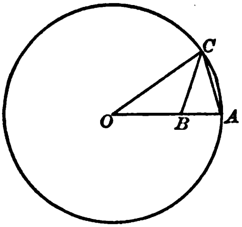
2. If *AC* denotes the side of a regular decagon, and *O* the center of the circumscribed circle, how many degrees in each angle of the triangle?

3. If, in the figure, *CB* be drawn bisecting angle *C*, how many degrees in each angle of $\triangle ABC$? Prove that $OB = CB = AC$.

4. Prove that $\triangle ABC$ is similar to $\triangle OAC$, and show that $\frac{OA}{AC} = \frac{AC}{AB}$. Show that $\frac{OA}{OB} = \frac{OB}{AB}$, and compare with Sec. 337.

PROPOSITION V. THEOREM

423. *To inscribe a regular decagon in a given circle.*



Given the circle with center O.

To inscribe a regular decagon in the circle.

Construction. 1. Divide any radius OA of the given circle in extreme and mean ratio at B, OB being the longer segment.

2. With A as center, and OB as radius, draw an arc cutting the circle at C. AC is a side of the regular inscribed decagon.

Proof. 1. $\frac{OA}{OB}=\frac{OB}{BA}$, step (1) of construction.

2. $\therefore \frac{OA}{AC}=\frac{AC}{BA}$, applying step (2) of Const. Sec. 78, I

3. $\angle A$ is common, therefore $\triangle$ OAC and ABC are similar. Sec. 316

4. $\therefore \angle AOC = \angle ACB$.

5. Since $\triangle$ OAC is isosceles, with equal angles A and C, $\triangle$ ABC is isosceles with equal angles A and B. Sec. 308

6. $\therefore BC = AC$, which equals OB by construction.

7. $\therefore \triangle$ OBC is isosceles, and $\angle BOC = \angle OCB$.

8. $\therefore \angle OCA = 2 \angle BOC$, step (4).

9. $\therefore$ in isosceles $\triangle$ AOC, $\angle AOC = \frac{1}{5}$ st. $\angle$, or $\frac{1}{10}$ of 360°.

10. But in a regular decagon, the central angle subtended by one side is $\frac{1}{10}$ of 360°, or 36°.

11. Hence, AC is one side of a regular decagon. The remaining sides may now be drawn as in step (2) of the construction.

424. Cor. 1. *To inscribe a regular pentagon in a given circle.*

Inscribe a regular decagon and connect the successive alternate vertices.

425. Cor. 2. *To inscribe regular polygons of 20, 40, 80, etc., sides in a given circle.*

Bisect the arcs subtended by the sides of a regular inscribed decagon and connect with the adjacent vertices. Treat similarly the resulting polygon.

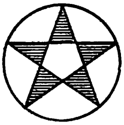
426. Cor. 3. *To inscribe a regular polygon of 15 sides in a given circle.*

Find the difference between the arcs subtended by a side of a regular inscribed hexagon and a side of a regular inscribed decagon. The chord of this arc is a side of a regular polygon of 15 sides. Why?

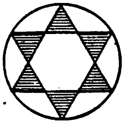
EXERCISES

Show how to construct:

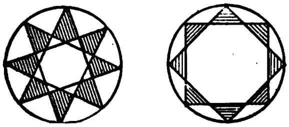
1. A five-pointed star.



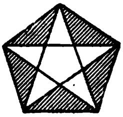
2. A six-pointed star.



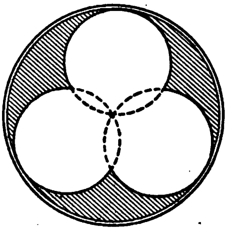
3. An eight-pointed star in two different ways.



4. Explain how to construct this ornamental design.



5. Plan a construction for this standard design of a circular window.



In the figure of Exercise 1:

6. Show that the perpendiculars from the center of the circle to the sides of the star pass through the vertices of the star.

7. By what arcs are the angles at the vertices of the star measured? Show that these angles are all equal.

8. By what arcs are the angles of the small pentagon measured? Show that it is equiangular.

9. Show that the small shaded triangles are congruent, and hence that the small pentagon is regular.

HISTORICAL NOTE

GAUSS

Propositions Sec. 414 to Sec. 426, with their corollaries, show how to construct certain regular polygons. The following table gives all polygons of fewer than 100 sides, which can be constructed in this way:

BASED UPON THE:	FORMULA	NUMBER OF SIDES
Regular triangle, Sec. 418	$2^n \cdot 3$	3, 6, 12, 24, 48, 96
Square, Sec. 414	2^n	4, 8, 16, 32, 64
Regular pentagon, Sec. 424	$2^n \cdot 5$	5, 10, 20, 40, 80
Regular pentadecagon, Sec. 426	$2^n \cdot 15$	15, 30, 60

These constructions were known to the Greeks. Book IV of Euclid's Elements gives the constructions for the regular triangle, the square, the regular pentagon, hexagon, decagon, and pentadecagon, but these were not original with Euclid. The construction of the regular hexagon was known to the Babylonians, and the pentagon and decagon were studied by the Pythagoreans. Among the lore that scholars have collected about the Pythagorean School is the story of the disciple Hippasus. He is said to have constructed the regular pentagon and to have been drowned for claiming the discovery instead of ascribing it to Pythagoras, as he should have done under the rules of the order.

But, neither the Greeks nor European scholars, until recent times, could extend the above list of polygons. Neither did they know the nature of the limitation imposed by the use of the straight-edge and compasses. In 1801 Gauss showed that the regular polygon of 17 sides can be constructed with ruler and compasses; and generally, that it is possible thus to construct any polygon if the number of its sides is prime and of the form $2^n + 1$. The proof is somewhat difficult. Probably the simplest discussion of the whole subject hitherto published is that of Dickson,[1] in a monograph entitled: "Regular Polygons; Constructions with Ruler and Compasses."

KARL FRIEDRICH GAUSS

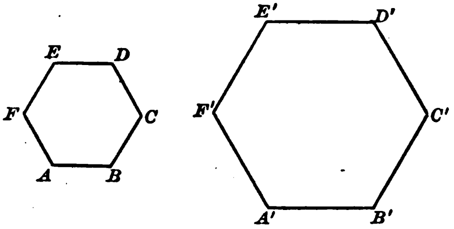
Karl Friedrich Gauss was born at Brunswick, Germany, in 1777. His father was a mason and took little interest in his son's education, but the boy's wonderful genius for numbers attracted the attention of his teachers, who induced the Duke of Brunswick to send young Gauss to a preparatory school. He entered the University of Göttingen in 1795 and soon made discoveries in the properties of numbers that won for him high rank among mathematicians. On the appearance of his great work, "Disquisitiones Arithmeticæ," published in 1801 when Gauss was only twenty-four years old, his contemporary, Laplace, declared Gauss to be the greatest mathematician of all Europe. Gauss died in 1855 after a life devoted to mathematics, enriching all its branches, including its applications in astronomy and physics, with the lasting products of his wonderful genius. With Weber, he invented the electric telegraph. Gauss has been worthily called: *Princeps Mathematicorum*, the Prince of Mathematicians.

---

[1] In "Monographs, etc." See p. 213.

PROPOSITION VI. THEOREM

427. *Any two regular polygons of the same number of sides are similar.*



Given two regular polygons, $A \cdots F$, $A' \cdots F'$, of the same number of sides.

To prove that they are similar.

Proof. It is sufficient to show:

(1) that corresponding angles are equal.

(2) That the ratios of corresponding sides are equal.

1. The size of the angles of a regular polygon depends only upon the number of sides of the polygon. As this number is the same for both polygons, the angles of the one polygon are equal to those of the other.

2. Since the polygons are regular, we have $AB = BC$, also $A'B' = B'C'$.

$\therefore \frac{AB}{A'B'} = \frac{BC}{B'C'}$, and similarly for all corresponding sides.

Consequently, the polygons are similar. Sec. 395. Q.E.D.

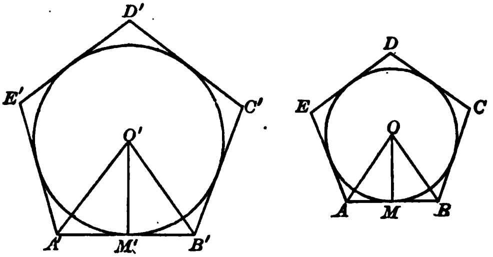
EXERCISES

1. Each angle of a certain regular polygon is 160°. How many sides has the polygon?

2. Show that the diagonals of a regular pentagon are equal.

PROPOSITION VII. THEOREM

428. *The radii of the inscribed circles of two regular polygons of the same number of sides have the same ratio as any two corresponding sides of the polygons.*



Given two regular polygons, $ABCDE$ and $A'B'C'D'E'$, of the same number of sides. Let O and O' be their centers.

To prove that $\frac{OM}{O'M'} = \frac{AB}{A'B'}$.

Proof. 1. $\triangle$ AMO and $A'M'O'$ are similar. Sec. 311

2. $\therefore \frac{OM}{O'M'} = \frac{AM}{A'M'} = \frac{2\,AM}{2\,A'M'} = \frac{AB}{A'B'}$. Q.E.D.

429. COR. 1. *The radii of two circumscribed circles of two regular polygons of the same number of sides, are in the same ratio as the corresponding sides of the polygons.*

430. COR. 2. *The perimeters of two regular polygons of the same number of sides have the same ratio as the radii of the inscribed circles, or as the radii of the circumscribed circles.*

For, in the figure above, the perimeters may be represented by nAB and $nA'B'$, where n is the number of sides.

$\therefore \frac{nAB}{nA'B'} = \frac{OM}{O'M'}$, or using r, r' for the radii, and P, P' for the perimeters, $\frac{r}{r'} = \frac{P}{P'}$. Similarly for the circumscribed circle.

431. Cor. 3. *The perimeters of regular polygons of the same number of sides are proportional to their apothems.*

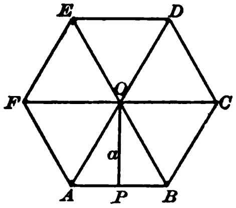
For, the radius of the inscribed circle of a regular polygon is the apothem of the polygon. Sec. 413.

Linear unit. The unit in which the lines of a figure are measured is called a **linear unit.**

Unit of area. Areas are measured in terms of the square of the linear unit used.

Proposition VIII. Theorem

432. *The area of a regular polygon is half the product of its perimeter and its apothem.*



Given the regular polygon P of n sides, having the apothem a and perimeter p.

To prove that its area $= \frac{1}{2}ap$.

Proof. 1. The lines OA, OB, etc., form n equal triangles. OAB, OBC, etc.

2. The area of $\triangle OAB$ is $\frac{1}{2}a \cdot AB$, of $\triangle OBC = \frac{1}{2}a \cdot BC$, and so on.

3. $\therefore P = \frac{1}{2}a \cdot AB + \frac{1}{2}a \cdot BC + \cdots = \frac{1}{2}a(AB + BC \cdots) = \frac{1}{2}ap$.

Q.E.D.

433. Cor. 1. *The areas of two regular polygons of the same number of sides have the same ratio as the squares of any two* *corresponding sides, or of the squares of the radii of: (a) the inscribed circles, (b) the circumscribed circles.*

Apply Sec. 427 and Sec. 379.

434. Cor. 2. *The areas of regular polygons of the same number of sides are proportional to the squares on their apothems.*

EXERCISES

In the following exercises rationalize the denominators in the results, but do not extract inexact roots, unless directed to do so.

In a circle of radius r, find an expression for:

1. The side of the inscribed square.

2. The apothem of the inscribed square.

3. The area of the inscribed square.

4. The side, apothem, and area of the circumscribed square.

5. The ratio of the area of the circumscribed square to that of the inscribed square.

6. The side, apothem, and area of the inscribed regular hexagon.

7. The side, apothem, and area of the circumscribed regular hexagon.

8. The ratio of the area of the hexagon of Ex. 7 to that of the hexagon of Ex. 6.

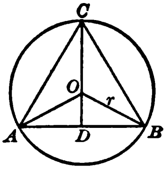
9. If ABC is an inscribed equilateral triangle show that:

(1) $\angle BOC = 120°.$

(2) $\angle BOD = 60°.$

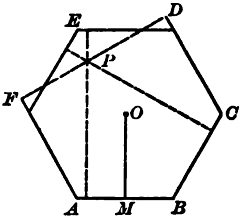
(3) $\dfrac{BD}{r} = \dfrac{\sqrt{3}}{2}.$

(4) Find the expressions for the apothem, side, and area of $\triangle ABC$.



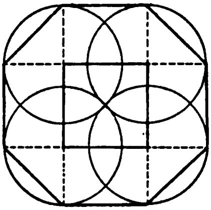
10. Find an expression for the apothem of the circumscribed equilateral triangle; for its area, its perimeter, its side; for the ratio of its area to that of the inscribed equilateral triangle.

11. If from any point within a regular hexagon perpendiculars be dropped to the six sides, produced if necessary, the sum of these perpendiculars is six times the apothem.



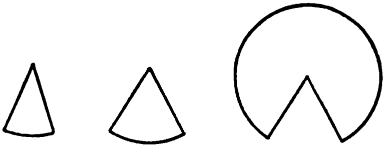
Suggestion. Connect the given point P with all the vertices, and compare the sum of the areas of all the triangles with the area of the polygon expressed in terms of OM and the perimeter p. Sec. 432.

12. If the sides of a square be prolonged to meet circles drawn about the vertices as centers and with radii equal to half the diagonal, show that the lines, joining the extremities of these lines in order form a regular octagon.



13. If the side of the square is s, show that the area of the octagon is $2s^2(1+\sqrt{2})$.

435. Sector. The figure formed by two radii of a circle and the intercepted arc is called the **sector** of the circle.



On comparing a sector with a triangle, it appears that they differ only in that the triangle has a straight base where the sector has a curved side. It accordingly seems natural to think that the area of a sector is half the product of its base (the arc) and its altitude (the radius). It can be proved that this is correct, but the proof is difficult, and we therefore accept the following rule without formal proof:

The area of a sector is half the product of its arc and the radius.

436. *The ratio of the circumference of a circle to its diameter is denoted by the Greek letter π (pronounced pī).*

Denoting the circumference by C, we have,

$$\frac{C}{D} = \pi,$$

or $$C = \pi D,$$

and since $D = 2r$, $$C = 2\pi r.$$

The value of π lies between 3 and 4. It is not a rational fraction, but its value can be calculated by methods of higher mathematics with any desired degree of approximation. (An elementary method is given in the supplement.) The value of π within one-billionth is:

$$3.141592645.$$

The value within one ten-thousandth, or

$$3.1416,$$

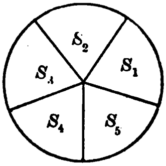
is sufficient for most computations, in fact, the value within $\frac{1}{7}$,

or $$\pi = \tfrac{22}{7},$$

is satisfactory in many practical applications and, for that reason, is sometimes called the *engineers' value.*

Proposition IX

437. *The area of a circle is π times the square of the radius.*



Proof.* Divide the circle into several sectors. Then the area of each sector is half its arc times the radius. The sum of the areas of all the sectors is the area of the circle. The sum of the bases is the circumference. Hence,

$$\text{Area of circle} = \frac{2\pi r}{2} \cdot r, = \pi r^2.$$

---

[*] On pages 299–303, will be found a treatment of this proposition according to the method of limits in the customary form, which may be studied in this connection, if thought best by the teacher.

HISTORICAL NOTE

ARCHIMEDES

Of the Greek mathematicians, Archimedes gave the best approximation to the value of π. By methods quite similar to those used in the present work, he established in his book on "The Measurement of the Circle," that π is less than $3\frac{1}{7}$ and greater than $3\frac{10}{71}$.

Archimedes lived at Syracuse in the third century B.C., and as a mathematician and physicist was one of the very greatest of antiquity. His versatility and the importance of his discoveries gave him rank among the scientists of his day analogous to that which Newton holds in modern times. Tradition has preserved many fascinating stories of his genius. Although of humble birth he is said to have won the admiration of King Hieron, and in return for the king's friendship Archimedes rendered many important services. The king, suspecting a fraud in the making of the royal crown, summoned Archimedes, who tested its purity by the now familiar method of specific gravity. Archimedes took part in the defense of Syracuse against the Roman invasion. He constructed many engines of war, and by a system of reflecting mirrors, so focused the sun's

ARCHIMEDES OF SYRACUSE

rays as to set fire to the Roman galleys. When praised for launching a ship by a system of pulleys, he replied, "With a lever I could lift the world, if there were a place for a fulcrum." Many of the propositions of solid geometry also, especially those relating to the sphere, were discovered by him. He himself valued most highly the theorem which establishes the ratio of the sphere to the circumscribed cylinder.

Archimedes perished in the ruthless slaughter that followed the fall of Syracuse (212 B.C.). We are told that he was studying some problem from diagrams drawn upon the sand, which in those days served instead of a blackboard, when a Roman soldier approached. Archimedes, absorbed in his study, called out "Do not destroy my circles." The Roman, incensed that one of the vanquished should venture to command him, slew the scientist with his spear. The commanding general, Marcellus, deeply regretted this needless sacrifice of so great a genius, and ordered erected to his memory a tomb on which was engraved the cylinder and sphere in accordance with an oft-expressed wish of Archimedes. It is likewise a strange fate that this tomb should have been lost for a century and rediscovered by the great Roman, Cicero, who identified the neglected grave by its unique inscription.

EXERCISES

PROBLEMS OF COMPUTATION

Using $3\frac{1}{7}$ for π, find the circumferences of the circles having the following radii:

1. 7 in.
2. 14 ft.
3. 21 yd.
4. 4.2 in.
5. $5\frac{3}{5}$ in.
6. .28 ft.
7. 1 ft. 9 in.
8. 2 ft. 4 in.

Using 3.1416 for π, find the circumferences of circles with diameters as follows:

9. 3 in.
10. 1 ft.
11. 2.5 yd.
12. .25 ft.
13. $3\frac{5}{8}$ in.
14. $5\frac{3}{4}$ ft.
15. 1 ft. 6 in.
16. 2 yd. 1 ft.

Using $\pi = 3.1416$, find the radii of circles with the following circumferences:

17. 8π.
18. 1.5π.
19. 3.1416 in.
20. 15.708 ft.
21. 31.416 yd.
22. 47.124 ft.

Using $\pi = 3.1416$, find the diameters of circles of the following circumferences:

23. $\frac{3}{4}\pi$ in. **25.** 47.124 in. **27.** 125.664 ft.
24. $.9\pi$ ft. **26.** $2\pi r$. **28.** 204.204 in.

Using $\pi = 3.1416$, find the areas of circles with radii as follows:

29. 4 in. **31.** $2x$ in. **33.** 2.5 ft. **35.** 1 ft. 6 in.
30. 3 ft. **32.** 3π yd. **34.** $6\frac{1}{2}$ ft. **36.** 3 ft. 5 in.

Using $\pi = 3.1416$, find the areas of circles with diameters as follows:

37. 14 ft. **39.** 8π in. **41.** $4\frac{2}{3}$ yd. **43.** 4 ft. 8 in.
38. 3.8 in. **40.** $20a$ ft. **42.** $12\frac{5}{8}$ ft. **44.** 2 ft. 4 in.

Using $\pi = 3.1416$, find the areas of circles with circumferences as follows:

45. 94.248 in. **47.** 376.992 ft. **49.** 15.708 rd.
46. 16π in. **48.** 392.70 yd. **50.** 47.124 ft.

Find the radii of circles containing areas as follows:

51. 314.16 sq. ft. **53.** $.09\pi$ sq. rd. **55.** π sq. ft.
52. 201.0624 sq. in. **54.** 64π sq. ft. **56.** πR^2.

PROBLEMS OF CONSTRUCTION

1. To circumscribe a square about a given circle.

2. To inscribe in a circle a regular octagon.

3. To inscribe in a circle a regular polygon of sixteen sides.

4. On a given line as a side, to construct a regular hexagon.

5. On a given line as a side, to construct a regular octagon.

6. To circumscribe a regular pentagon about a given circle.

To divide an angle of 360° into:

7. 3 equal parts. **9.** 5 equal parts. **11.** 8 equal parts.
8. 4 equal parts. **10.** 6 equal parts. **12.** 10 equal parts.

To divide an angle of 180° into:

13. 3 equal parts. **15.** 5 equal parts. **17.** 8 equal parts.

14. 4 equal parts. **16.** 6 equal parts. **18.** 10 equal parts.

19. On a given line as side, to construct a regular pentagon.

Suggestion. Construct an arbitrary regular pentagon. The fourth proportional to its side, its apothem and the given side, in this order, will be the apothem of the desired pentagon.

20. To divide the area contained in a given circle into two equal parts by a concentric circle.

21. To draw lines across the corners of a given square so as to make a regular octagon.

22. To construct a square equivalent to a given regular octagon.

23. To construct a circle whose area shall have a given ratio to that of a given circle.

24. To inscribe in a given circle a regular polygon similar to a given regular polygon.

Applied Problems

1. If water or other liquid flows through a cylindrical pipe, the area of whose cross section is 1 sq. in., at the rate of 5 in. per second, how many cubic inches flow through in a minute?

2. Assuming that the rate of flow is the same in all pipes of the same waterworks system, if a 2-in. pipe (*i.e.* a pipe having an inner diameter of 2 in.) furnishes c cu. in. of water per second, how many cubic inches will a 5-in. pipe furnish?

3. A certain factory is supplied with water by a 4-in. main. Owing to enlargements, it needs to double its water supply. If the pipes available for the mains are to be had only in even inch and half-inch sizes, what is the smallest size of main that will supply the desired amount of water?

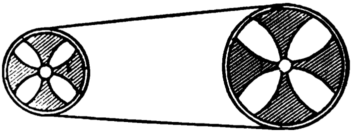
4. If the weight that a circular pillar of given material will support is proportional to the area of its cross-section, which will support the heavier weight, other conditions being equal, six 8-in. pillars or eight 6-in. pillars?

5. If 5 d-in. pillars support the same weight as d 10-in. pillars, find d.

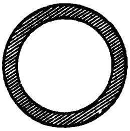
6. The pressure required to force water through a pipe is largely used in overcoming friction against the walls of the pipe. If a 2-in. pipe is replaced by a 6-in. pipe, in what ratio is the surface of the pipe (and consequently the amount of friction) increased?

7. How many revolutions does the wheel of an automobile make in going a mile, if its diameter is 30 in.?

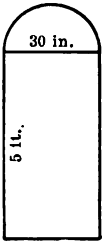
8. A belt connects two wheels, one 18-in. in diameter and the other 12 in. How many revolutions does the smaller wheel make while the larger makes 100?



9. An asbestos gasket used to pack the manhole of a steam boiler is in the form of a ring 21 in. in outside diameter and 3 in. wide. Find its area.

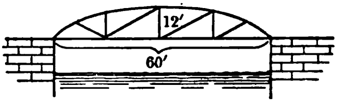


10. Find the area of the window in the figure, including the semicircular top.

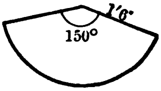


11. Find the diameter of a drain tile whose cross-section shall be equal to the sum of the cross-sections of two tiles 3 in. and 5 in. in diameter respectively.

12. A bridge 60 ft. long is supported by an arch in the form of a circular arc 12 ft. above the bridge at its highest point. Find the radius of the arc.



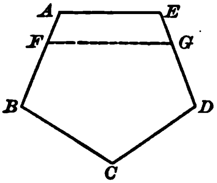
13. A piece of metal to be rolled into a cone is in the form of a sector of a circle. Its radius is 1 ft. 6 in. long, and its angle at the vertex is 150°. Find the area of the sector in square inches to 4 decimal places.



THEOREMS

1. Is it possible to construct an equiangular pentagon that is not regular?

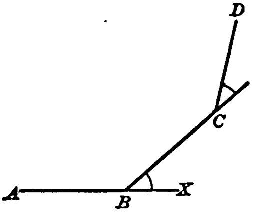
Suggestion. Let $A \cdots E$ be a regular pentagon and let $FG \parallel AE$. Is $FBCDG$ an equiangular pentagon? Is it regular?



2. If the sides of an equiangular hexagon are numbered 1 to 6 in order, show that sides Nos. 1 and 4 are parallel; likewise Nos. 2 and 5, 3 and 6.

3. If the sides of an equiangular polygon of $2n$ sides are numbered $1 \cdots 2n$, show that sides Nos. 1 and $n+1$, 2 and $n+2$, etc., are parallel.

Suggestion. Let $ABCD$ be a portion of a polygon of $2n$ sides. Imagine a man walking around the polygon from A to B. At B he turns through the angle XBC, or $\frac{180°}{n}$. At C he turns through the same angle, and on CD he is going at an angle of $2 \cdot \frac{180°}{n}$ with AB, and so at each corner he turns through an angle of $\frac{180°}{n}$. If the total amount of his turning is ever exactly 180°, then the side on which that happens will be parallel to the side AB, and he will pass along that side in the opposite direction from that in which he passed along AB. But as he turns $\frac{180°}{n}$ each turn, the total amount of turning will be exactly 180° when he has passed n corners, or the nth side after AB will be parallel to AB; or, if we make AB as the first side, BC as the second, etc., the $(n+1)$st side is parallel to the first.



4. Show that no two sides of an equiangular polygon of $2n + 1$ sides are parallel.

Suggestion. Proceeding as in Exercise 3, the amount of turning at each vertex is $\frac{360°}{2n+1}$. The question is: Is there a number x, of turns such that

$$\frac{x \cdot 360°}{2n+1} = 180°,$$

or,

$$\frac{2x}{2n+1} = 1.$$

In the nature of the problems x and n are integers. Hence, we have here an even integer divided by an odd integer, and this cannot have the quotient unity. Hence, no such number x exists, and no two sides of the polygon are parallel.

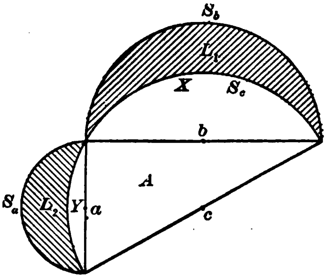
5. If semicircles be drawn on the three sides, a, b, c, of a right triangle as indicated, the sum of the areas of the moon-shaped figures formed (called lunes of Hippocrates) is equal to the area of the triangle.

Suggestion. Let S_a, S_b, S_c denote areas of semicircles on a, b, c.

Then, $S_c = \frac{\pi c^2}{8}$.

$$S_a = \frac{\pi a^2}{8}.$$

$$S_b = \frac{\pi b^2}{8}.$$



Let A denote the area of $\triangle ABC$, X and Y the areas of the small segments, and L_1 and L_2 the areas of the two lunes indicated.

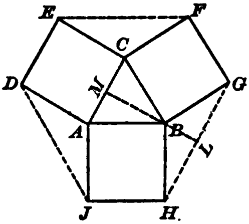
Then,

$$S_c = A + X + Y.$$
$$S_b = X + L_1.$$
$$S_a = Y + L_2.$$
$$\therefore S_a + S_b - (L_1 + L_2) = X + Y.$$
$$\therefore S_c = A + S_a + S_b - (L_1 + L_2),$$

or,

$$\begin{aligned} L_1 + L_2 &= A + S_a + S_b - S_c \\ &= A + \frac{\pi}{8}(a^2 + b^2 - c^2) \\ &= A + \frac{\pi}{8} \cdot 0 \\ &= A. \end{aligned}$$

6. If the pairs of tangents from two external points include equal angles, show that their chords of contact are equal.



7. If on the sides of an equilateral triangle squares be erected and the vertices joined as indicated, show that $HG \parallel AC$, $JD \parallel BC$, $EF \parallel AB$. Show that $\triangle HLB = \triangle ABM$. Show that the three triangles corresponding to $\triangle CEF$ are equivalent. If $AC = b$, show that area $DEFGHJ$ is $b^2(3 + \sqrt{3})$.

HISTORICAL NOTE

Pi

No single fact of geometry has a more fascinating and significant history than the ratio of the circumference of a circle to its diameter. The earliest guess at this value was doubtless 3, for it appears often in the records of ancient peoples. The Hebrews knew it as such, for the twenty-third verse of the seventh chapter of First Kings reads: "He made a molten sea 10 cubits from the one brim to the other . . . and a line of 30 cubits did compass it round about." The Babylonians likewise used 3; but the Egyptians, versed in the art of measuring land, found a nearer value. Ahmes, an Egyptian teacher, about 1700 B.C. gives in the famous Rhind Papyrus, now in the possession of the British Museum, $(\frac{8}{9})^2d^2$ as the area of a circle; this formula gives $\pi = 3.16+$. The great Greek Archimedes (about 250 B.C.), by calculating the perimeters of regular polygons, found this ratio to lie between $3\frac{1}{7}$ and $3\frac{10}{71}$. Chinese mathematicians, about 500 A.D., arrived at a closer value by locating this ratio between 3.1415926 and 3.1415927. The Hindu priests of India, at the same time, used $\frac{62832}{20000}$, or 3.1416. The mathematicians of the Middle Ages could do no better, but Vieta, in the sixteenth century, computed its value to 9 decimal places.

The Greek letter π, the first letter of the word *periphery*, was not employed by the Greeks to denote the ratio of the circumference of a circle to its diameter, but was introduced by an Englishman, William Jones, in the eighteenth century.

After Vieta, mathematicians sought the value of π with great zeal. In the seventeenth century, the Dutch calculated it to 35 decimal places.

The scholars of the eighteenth century found 200 places, and in the nineteenth century Richter computed 500 places and Shanks 707.

That mathematicians were willing to undergo the prodigious toil necessary to produce these results need not be ascribed to mere curiosity or to the spirit of competition, but may be attributed to appreciation of the theoretic importance of an exact value for π. Scholars of all times had sought to measure exactly the length of a circle and its area. If $c = 2\pi r$ were known exactly, πr^2 would be known. Furthermore, πr^2 could be expressed as a rectangle which, by Sec. 385, could be replaced by an equivalent square. In other words, if π were known exactly, the famous problem "To square the circle" would be solved.

It would be unfair to say that the work spent on the calculation of π was wasted, but results established in the last half of the nineteenth century show that all attempts to find the value of π as an exact decimal or to find a method of constructing, by ruler and compasses, a square equivalent to a given circle, were in the very nature of things predestined to be futile. The strict proof of this assertion was first given by Lindemann in 1882. He proved much more than what is asserted above, by showing that π is not a surd of any description, but belongs to a class of numbers that are not roots of any algebraic equation whatever with rational coefficients; such numbers are known as *transcendental numbers*, that is, numbers lying outside the domain of algebra. An account of this is given by Smith in a monograph[1] entitled, "The History and Transcendence of π."

---

[1] In "Monographs, etc.," see p. 213.

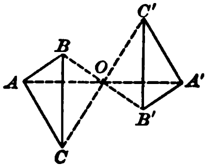
SUPPLEMENT

I. SYMMETRY

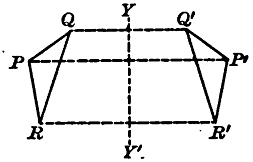
438. Symmetric Points. Two points are said to be *symmetric with respect to a point* if this point bisects the straight line connecting the two given points. The bisecting point is called the *center of symmetry.*

439. Two points are said to be *symmetric with respect to a straight line* if the straight line bisects at right angles the line connecting the two points. The bisecting straight line is called the *axis of symmetry.*

440. Symmetric Figures. Two figures are said to be symmetric (with respect to a center or an axis) if each figure is the locus of the points symmetric to the points of the other figure.



O IS THE CENTER OF SYMMETRY



YY' IS THE AXIS OF SYMMETRY

If two figures are symmetric with respect to an axis, one can be brought to coincide with the other by folding the plane along the axis of symmetry. If two figures are symmetric with respect to a point, one can be brought to coincide with the other by moving it around in the plane.

441. A single figure is said to be symmetric (with respect to a center or an axis) if the symmetric point of every point of the figure is likewise a point of the figure.

EXERCISES

1. The center of a circle is the center of symmetry of the circle.

2. Any diameter of a circle is an axis of symmetry.

3. Either diagonal of a square is an axis of symmetry of the square.

4. The intersection of the diagonals of a square is a center of symmetry.

5. The center of a regular hexagon is a center of symmetry.

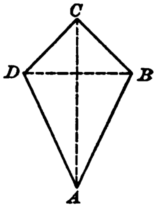
6. The diagonals of a rhombus are axes of symmetry of the rhombus.

7. An equilateral triangle has three axes of symmetry.

8. The line joining an external point to the center of a circle is an axis of symmetry for the tangents to the circle from the given point.

9. The line of centers of two circles is the axis of symmetry for the pairs of the common tangents to the circles.

10. A quadrilateral with only two pairs of adjacent sides equal has one axis of symmetry. In the figure $AB = AD$ and $BC = DC$; show that $\triangle ABC$ is congruent to $\triangle ADC$ and that AC is an axis of symmetry of the quadrilateral. By what property of symmetry is $DB \perp AC$?

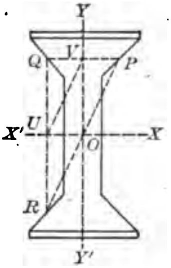


Quadrilaterals with an axis of symmetry have been called "Kites" by the English mathematician Sylvester, who studied many properties of symmetric figures.

11. Two figures symmetric either with respect to an axis or with respect to a center are congruent.

Does a line through the center of symmetry of a figure always divide a figure into two congruent parts?

12. If a figure has two axes of symmetry at right angles, the intersection of these axes is the center of symmetry of the figure. Let P be any point of the given figure and XX', YY' be the axes of symmetry making right angles at O. Let Q be the symmetric point to P with respect to YY', and R the symmetric point to Q with respect to XX'. Connect P with R and V with U. In $\triangle PQR$, $XX' \parallel PQ$, Sec. 126, and passes through the mid-point of QR. $\therefore$ it bisects PR. Similarly, YY' bisects PR. $\therefore$ the intersection of the axes of symmetry is the mid-point of PR and P is symmetric to R with respect to O as a center. Sec. 438. Similarly for any other point.



II. Maxima and Minima

442. Maxima and Minima. Among the set of values that a variable geometric magnitude may have, the greatest is called the *maximum* and the least the *minimum*.

For example, of all chords that can be drawn in a given circle, the diameter is the maximum.

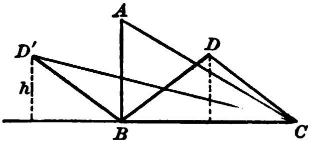
Of all segments that can be drawn from a point to a line, the perpendicular is the minimum.

443. Isoperimetric Figures. Figures whose perimeters are equal are called *isoperimetric* figures.

For example, a square of side 6 in. and an equilateral triangle of side 8 in. Each has a perimeter of 24 in., and they are, therefore, isoperimetric figures. Also a circle of radius 2 ft. and a square of side π ft. are isoperimetric figures.

PROPOSITION I. THEOREM

444. *Of all triangles having two given sides, that in which these sides include a right angle has the maximum area.*



Given ⚠ ABC, $D'BC$ and DBC, with $\angle ABC$ a right angle, and $DB = AB = BD'$.

To prove that area ABC is the maximum.

Proof. 1. Draw the altitude from D. It is less than AB. Secs. 189, 210

2. $\therefore$ area $DBC <$ area ABC. Sec. 358

3. Similarly area $D'BC <$ area ABC.

4. Hence, the triangle ABC has the maximum area Q.E.D.

445. COR. 1. *Conversely, if two sides of a triangle are given and if the area of the triangle is a maximum, the given sides include a right angle.*

446. COR. 2. *Of all parallelograms having the same given sides, the one that is rectangular has the maximum area, and conversely.*

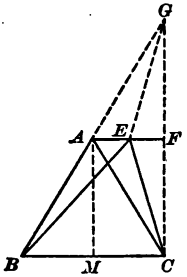
EXERCISES

1. A triangular fly swatter has two of its sides each 4 inches long. How must the swatter be shaped to have the largest possible area?

2. In the triangle ABC, C is a right angle, $AB = 2AC$, D is the mid-point of AB and E that of CB. Show that the area ACE is greater than area CDE.

PROPOSITION II. THEOREM

447. *Of all equivalent triangles having the same base, that which is isosceles has the minimum perimeter.*



Given equivalent $\triangle$ ABC and BEC with the same base BC. Suppose that $AB = AC$ and that BE and EC are unequal.

To prove that $AB + BC + CA < EB + BC + CE$.

Proof. 1. Draw at C a perpendicular to BC, and call its intersection with AB extended G. Connect E with G and extend AE to meet CG at F. $AF \perp GC$. Why?

2. $AF = MC = BM$. Why?

3. $\angle ABM = \angle GAF$.

4. $\triangle AMB$ is congruent to $\triangle GFA$. Why?

5. $\therefore AB = AG$.

6. $EG = EC$, and $AG = AC$. Sec. 106

7. $\therefore BA + AC = BG$, and $BE + EC = BE + EG$.

8. $\therefore BA + AC < BE + EC$. Why?

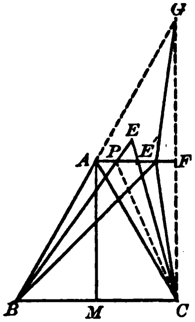
9. $\therefore AB + BC + AC < EB + BC + CE$.

10. $\therefore$ the perimeter of $\triangle$ ABC is minimum. Q.E.D.

448. COROLLARY. *Of all equivalent triangles the equilateral one has the minimum perimeter.*

PROPOSITION III. THEOREM

449. *Of all isoperimetric triangles of the same base, the isosceles triangle has the maximum area.*



Given an isosceles $\triangle ABC$ and any other $\triangle EBC$ having the same base BC and the same perimeter as $\triangle ABC$.

To prove that area $ABC >$ area EBC.

Proof. 1. Draw at C a line $\perp BC$ and meeting the extension of BA at G. Draw $AF \parallel BC$.

2. $\angle ACG = \angle AGC$. Complements of equal $\angle$s.

3. $\therefore GA = AC = AB$. Why?

4. $AF \perp GC$ at its mid-point F. Why?

5. If we show that E cannot lie on AF or above it, we shall have shown that area $ABC >$ area EBC. Why?

6. $AB + AC = EB + EC$. Given

7. E cannot lie on AF, as at E', For

$$BE' + E'C = BE' + E'G$$

and $$BE' + E'G > BA + AG,$$

but $$BA + AG = BA + AC. \qquad \text{Step (3)}$$

$\therefore BE' + E'C > BA + AC.$ contrary to (6)

8. E cannot lie above AF. For

$$BE + EC > BP + PC \qquad \text{Sec. 25, 1}$$

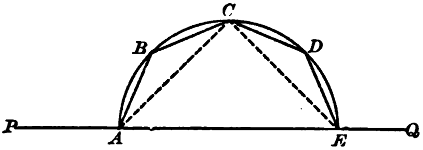
and $BP + PC > BA + AC.$ as in step (7)

$\therefore BE + EC > BA + AC.$ contrary to (6)

9. Hence, E falls below AF, and triangle ABC has the maximum area. Q.E.D.

PROPOSITION IV. THEOREM

450. *Of all polygons with all sides but one equal respectively to given lines, the polygon of maximum area can be inscribed in a semicircle on the undetermined side as diameter.*



Given. $ABCDE$, the polygon of maximum area having sides equal to lines AB, BC, CD and DE.

To Prove that $ABCDE$ can be inscribed in the semicircle having AE as diameter.

Proof. 1. Assume that any vertex, as C, does not lie on the semicircle of diameter AE. Connect C with A and with E.

2. Area ACE must be as large as possible in order that the polygon may have the maximum area, for area ACE can be changed by sliding the ends A and E of CA and CE along the line PQ, C acting as a hinge, while the other parts of the polygon, ABC and CDE, remain constant.

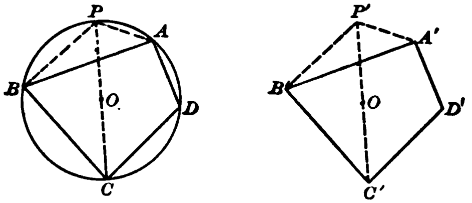
3. But, area ACE is maximum when ACE is a right angle. Sec. 444

4. $\therefore C$ is on a semicircle whose diameter is AE. Why?

5. Thus, the final side AE of the maximum polygon must subtend a right angle at each of the vertices other than A and E; hence the polygon $ABCDE$ is inscriptible in a semicircle having the diameter AE.

PROPOSITION V. THEOREM

451. *Of all polygons that have their sides equal respectively, the one that is inscriptible in a circle has the maximum area.*



Given polygon $ABCD$ inscriptible and polygon $A'B'C'D$ having sides respectively equal to those of $ABCD$, not inscriptible.

To prove that area $ABCD >$ area $A'B'C'D'$.

Proof. 1. From any vertex, as C, draw a diameter CP and draw PA and PB.

2. On $A'B'$ construct $\triangle P'A'B'$ congruent to $\triangle PAB$, and connect P' with C'.

3. Since $A'B'C'D'$ is not inscriptible, one or both of the parts $P'B'C'$, $P'A'D'C'$ are not inscriptible in the semicircles with diameter $P'C'$.

4. $\therefore PBC$ or $PADC$ or both must be greater, and neither can be less, than the corresponding parts of $P'B'C'D'A'$.

Sec. 450

5. $\therefore$ area $APBCD >$ area $A'P'B'C'D'$.

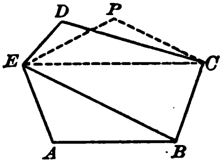
6. $\therefore$ area $ABCD >$ area $A'B'C'D'$. Step (2). Q.E.D.

EXERCISE

Using the proposition above, show that of all parallelograms having their corresponding sides equal, the rectangle has the largest area.

PROPOSITION VI. THEOREM

452. *Of all isoperimetric polygons of the same number of sides, that having the maximum area is equilateral.*



Given $ABCDE$ the polygon with the given perimeter and given sides, and having the maximum area.

To prove that $AB = BC = CD = DE$.

Proof. 1. Suppose any two adjacent sides, as ED and DC, to be unequal. Draw diagonal EC.

2. On EC construct an isosceles $\triangle PEC$, having

$$EP + PC = ED + DC.$$

3. Then area $PEC >$ area DEC. Sec. 449

4. $\therefore$ area $ABCPE >$ area $ABCDE$, while the perimeters remain the same.

5. $\therefore$ to suppose a pair of adjacent sides unequal leads to a contradiction.

6. But if all adjacent sides are equal, all sides are equal and the polygon of maximum area is equilateral.

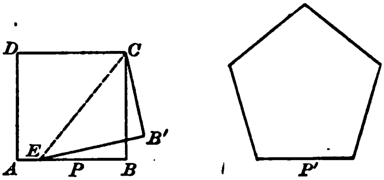
453. COROLLARY. *Of all isoperimetric polygons with a given number of sides, that having the maximum area is regular.*

EXERCISE

Using the proposition above, show that of all quadrilaterals having the same area, the square has the smallest perimeter.

PROPOSITION VII. THEOREM

454. *Of all isoperimetric regular polygons, that which has the greatest number of sides has the maximum area.*



Given a regular polygon P and an isoperimetric regular polygon P' with one side more.

To prove that area $P' >$ area P.

Proof. To prove the theorem it is sufficient to prove generally that in a succession of isoperimetric regular polygons each having one side more than its predecessor, each has a greater area than its predecessor.

1. Let P be a square and P' a regular isoperimetric pentagon. From C, any vertex of the square, draw a line to any point E in AB, making $\triangle EBC$. Turn this triangle over so that E falls at C, C at E and B at B'.

2. The polygon $AEB'CD$ is an irregular pentagon having the same perimeter and area as the square $ABCD$, or P.

3. But, area $P' >$ area $AEB'CD$. Sec. 453

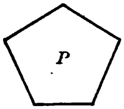
$\therefore$ Area $P' >$ area P.

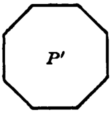
4. This method can be applied to any consecutive pair of regular polygons, for the line CE can always be drawn from any vertex to the second side from this vertex, and the triangle thus cut off can be turned over as above. An irregular polygon of one side more will result, equal in area and isoperimetric with the given polygon.

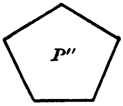
5. Hence in a succession of regular isoperimetric polygons each having one side more than its predecessor, each has a greater area than its predecessor. Q.E.D.

PROPOSITION VIII. THEOREM

455. *Of all regular polygons having a given area, that which has the greatest number of sides has the minimum perimeter.*







Given the regular polygons P and P' of the same area, P' having the greater number of sides.

To prove that the perimeter of $P >$ the perimeter of P'.

Proof. 1. Assume a third regular polygon P'' having the perimeter of P' and the same number of sides as P.

Let a be a side of P and a'' a side of P''.

2. Then, area $P' >$ area P'', Sec. 454

and area $P >$ area P'', since area $P = P''$. Given

3. But, area P : area $P'' = a^2 : a''^2$. Sec. 379

4. $\therefore\ a^2 > a''^2$. Step (2)

$\therefore\ a > a''$.

5. $\therefore$ the perimeter of $P >$ the perimeter of P''.

6. Finally, the perimeter of $P >$ the perimeter of P', since the perimeter of $P' =$ the perimeter of P''. Q.E.D.

EXERCISES

1. Compute the area of an equilateral triangle of perimeter 1 ft. Also of a square of perimeter 1 ft. Which is the larger?

2. Compute the perimeter of a square of area 1 sq. ft. Also compute the perimeter of an equilateral triangle of area 1 sq. ft. Compare these perimeters. What theorem is illustrated?

3. Show algebraically that of all equivalent rectangles the square has the minimum perimeter.

Suggestion. Let a be the side of the square, and b and c those of the rectangle. Then we have:

1. $a^2 = bc.$

And we wish to show that

2. $4a < 2b + 2c,$

3. or, by 1, that $2a < \frac{a^2 + c^2}{c},$

4. or that $2ac < a^2 + c^2,$

5. or that $0 < (a - c)^2.$

But we know that the last is true, since $a \neq c$.

4. To divide a given line into two such segments that their product shall be the maximum.

5. Of all triangles inscribed in a given circle, the equilateral one has the maximum perimeter.

6. To inscribe in a given segment a triangle whose perimeter is the maximum.

7. Given the base and opposite angle of a triangle to construct the triangle of maximum area.

8. From a point without a circle, to draw to the circle the line of maximum length. Also to draw the line of minimum length.

9. Of all triangles of given perimeter and a given base, the isosceles triangle has the maximum area.

10. To draw the parallelogram of maximum area with given diagonals.

11. To inscribe in a circle the rectangle of maximum area.

12. Required to find the minimum path from a point P to a given line l and thence to another given point Q on the same side of the line. See Ex. 36, p. 166.

13. Find the minimum chord that can be drawn through a given point in a circle.

III. SIGNED MAGNITUDES

456. Positive and Negative Numbers. In algebra we have become acquainted with magnitudes which may be taken in one of two senses or directions that are opposite in such a way that a unit of one sense offsets a unit of the other sense. We call one of the senses **positive**, and the other **negative**, and indicate them by the signs + and − respectively. The numbers measuring such magnitudes are called **positive and negative numbers.**

457. Directed Lines. We may fix a positive direction on a line, then the opposite direction is the negative direction, Fig. 1. For horizontal and vertical lines the directions are understood to be fixed as in Fig. 2, unless otherwise specified.

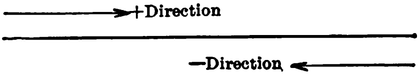


FIG. 1

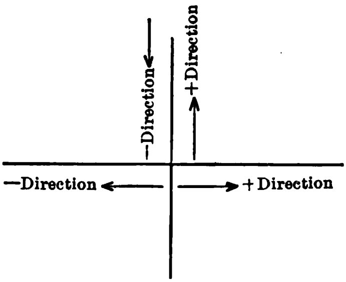
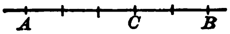


FIG. 2

458. Directed Segments. A segment is regarded as taken in the direction from the first point named (*the initial point*) to the second (*the terminal point*). If the segment is taken in the positive direction it is a *positive segment;* if taken in the negative direction it is a *negative segment.*

Thus, if AB in the figure of Sec. 459 is a positive segment, BA is a negative segment.

459. Addition of Directed Segments. Directed segments are added or subtracted by application of the property that a positive segment and a negative segment of the same absolute length offset each other. The terminal point of the first segment must be taken as the initial point of the second segment.



Thus, AB is a positive segment, 5 units long, and BC is a negative segment 2 units long. Hence

$$AB + BC = 5 + (-2) = 3.$$

We may think of the point moving from A to B, generating a positive segment (5 in. long), then from B to C, generating a negative segment (2 in. long). The total effect of the motion is a displacement from A to C. That is

$$AB + BC = AC.$$

In this example $5 + (-2) = 3.$

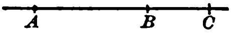
460. Subtraction of Directed Segments. To subtract a segment we add its negative segment.

Thus, $AB - CB = AB + BC$; or $5 - 2 = 5 + (-2)$.

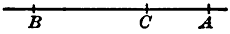
EXERCISES

Show that $AB + BC = AC$ for the following figures:

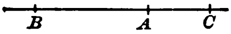
1.



2.



3.

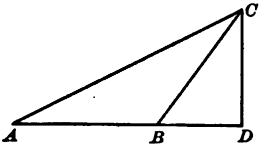


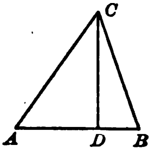
Calculate by use of points on a line as in the example above:

4. $2 + (-6)$.
5. $-4 + 5$.
6. $-4 + (-3)$.
7. $3 + 4$.
8. $5 - (2 + 3)$.
9. $-2 - 3$.
10. $(-6) - (-4)$.
11. $-5 + (-2)$.
12. $4 + (-5)$.

461. Application of Directed Segments. By the use of signed segments two or more theorems that must otherwise be proved separately, can be combined into a single theorem.

Thus, the two theorems of Secs. 375 and 376 can be combined into the following, provided we regard *BD* as a positive segment, if directed towards the right, and as a negative segment, if directed towards the left.





$$\overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2 + 2\,AB \cdot BD.$$

462. Positive and Negative Angles. An angle may be regarded as generated by the revolution of one of its sides about the vertex as a pivot, from one side of the angle (*the initial side*) to the other side (*the terminal side*).

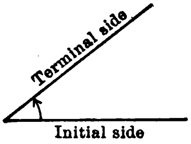


FIG. 1

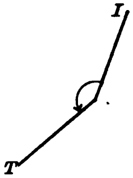


FIG. 2

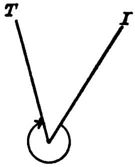


FIG. 3

463. The Positive Direction of revolution is that opposite to the direction in which the hands of a clock turn; the latter is the *negative direction.*

Thus, the angles in figures 1 and 2 above are positive angles, and the angle in figure 3 is a negative angle.

By continuing the revolution in either direction a positive or negative angle of any size is obtainable.

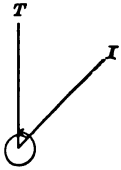
EXERCISES

Taking the initial side in various positions in each case, and using the notation of Sec. 462, construct roughly:

1. 40°. **3.** – 20°. **5.** 300°. **7.** – 200°.
2. 110°. **4.** – 80°. **6.** – 300°. **8.** 400°.

The solution is indicated in the figure.

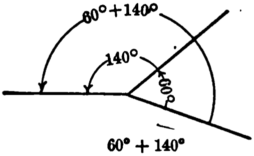
The motion need not cease when one complete revolution has been made, but may continue until the desired number of degrees has been turned. In this instance the revolving side advances 40° into the second revolution.



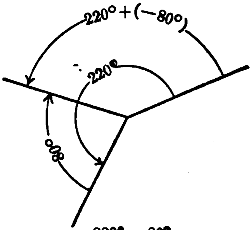
9. 500°. **11.** – 400°. **13.** – 1000°.
10. 620°. **12.** – 600°. **14.** 930°.

464. Addition of Directed Angles. Directed angles may be added in a manner quite similar to that in which directed segments are added. The vertex must be common and the terminal side of the first angle must be taken as the initial side of the second angle.

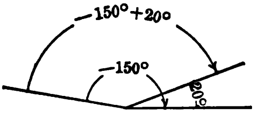
EXAMPLES:



60° + 140°



220° – 80°



– 150° + 20°

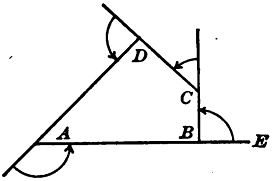
EXERCISES

Construct angles approximately equal to the following, taking the initial side of the first angle in various positions:

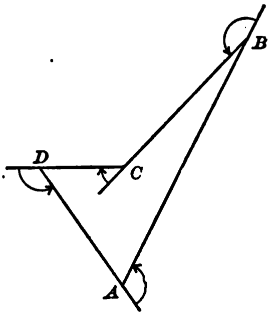
1. 110° + 150°. **2.** 100° + 200°. **3.** 60° + 60°.

4. $-50° + 30°$.
5. $-20° + (-50°)$.
6. $-10° + (-200°)$.
7. $60° + (-60°)$.
8. $400° + (-300°)$.
9. $200° + (-200°)$.
10. $-360° + 270°$.
11. $-130° + (-150°)$.

465. Application of Positive and Negative Angles. If the sides of any polygon are produced in order, and the angle less than a straight angle between the produced arm and the next following side, be defined as the *exterior angle* at that vertex, then we have seen (Sec. 176) that the sum of the exterior angles of any convex polygon is four right angles.

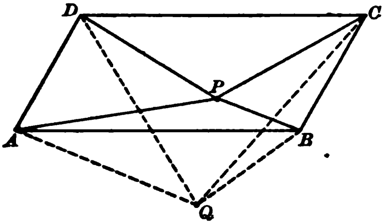


It is instructive for our present purposes to think of the proof as follows: Imagine a person walking forward around the polygon, say from A to B, C, D, and back to A. At each vertex he must turn through the exterior angle of that vertex in order to face in the direction he wishes to go next. At B, for example, he arrives facing in the direction BE, and must turn to face in the direction in which he would start on the second side; keeping on in this way, he has made one complete turn, or 4 right angles, when he reaches A, and faces B, ready to begin the second round. In this case the angles through which he turns are all positive angles.



Admitting negative angles, it can be seen in a similar manner that the theorem holds also for concave polygons. Thus in the figure at the left the exterior angles are as indicated, and the one at C is taken in the negative sense. The aggregate of all the angles is still one complete turn, or four right angles.

466. Positive and Negative Area. The areas of figures standing on one side of a common base line may be taken as positive and those standing on the other side as negative.



Given $\square\, ABCD$. If area of $\triangle\, APB$ is taken as positive, that of $\triangle\, AQB$ would be taken as negative. It can easily be proved that (for unsigned areas)

$$\text{area } PAB + \text{area } PDC = \tfrac{1}{2} \text{ area } ABCD,$$
$$\text{area } QDC - \text{area } ABQ = \text{area } ABCD.$$

But if we admit signed areas, and take as positive the areas of the triangles standing on AB or DC and having their vertices within the strip formed by AB and DC produced, and as negative the triangles that have their third vertex outside the strip, then we may combine all the above into the simple theorem:

If any point P *of the plane be joined by straight lines to the vertices of the parallelogram* ABCD, *the sum of the area of* PAB *and* PCD *is half the area of the parallelogram.*

467. Principle of Continuity. The above are illustrations of the fact, frequently found, that different theorems relating to figures of the same type can be combined into a single statement by the use of signed magnitudes. When all the figures of the type can be produced one from the other by a gradual change of the figure (as in all the illustrations above), it can very often be proved that the theorem continues to hold for all figures under a suitable interpretation of signs. The fact of this continuance has sometimes been called the **principle of continuity.**

IV. LIMITS

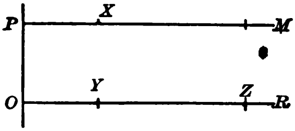
468. Constants and Variables. In geometry as in algebra a quantity regarded as having a fixed value is called a **constant**, and a quantity regarded as free to take different values is called a **variable.**

E.g., if in a discussion we consider all the regular polygons circumscribed about a circle of given radius, the radius of the circle or the apothem of the polygons is *constant*, and the area of the polygons is a *variable.*

NOTE.—The discussions below are not fully rigorous, but the treatment is that customarily accepted as sufficient by educators who desire to give the subject the maximum measure of formal treatment that is appropriate to a course in elementary geometry.

469. Limit. When the values of a variable approach a constant, in such a way that the difference between the constant and the value of the variable may become and remain numerically less than any assigned value, however small, the constant is called the **limit** of the variable.

470. Principle of Limits. *If, while approaching their respective limits, two variables remain equal, their limits are equal.*

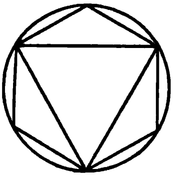


Suppose PX and OY to be two increasing variables, and suppose them to remain equal as they approach their limits PM and OR. To show that $PM = OR$, assume that these limits are not equal and let $OR > PM$, or $OZ = PM$. Then OY could eventually exceed OZ and hence exceed its equal PM. But PX can at most equal PM; hence, the supposition that $OR > PM$ leads to the conclusion that $OY > PX$, which is contrary to the hypothesis. Similarly, PM is not greater than OR.

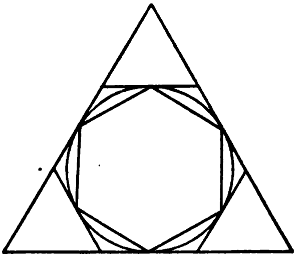
471. Cor. 1. *If* X *is a variable having the limit* L *and* a *is a constant, then* aL *is the limit of* aX.

472. Cor. 2. *If two variables* X *and* X′ *having the limits* L *and* L′ *have a constant ratio* r, *then* L *and* L′ *have the ratio* r.

V. THE MEASUREMENT OF THE CIRCLE



473. The Circle as a Limit. If we inscribe a regular triangle in a circle, and then inscribe a regular hexagon by taking the mid-points of the arcs for new vertices, as in the figure, the area of the hexagon is greater than that of the triangle but less than that of the circle. Also the perimeter of the hexagon is greater than that of the triangle, Sec. 80, IV, and less than that of the circle.



Similarly, if a regular triangle be circumscribed about a circle, its area is larger than that of the circle and its perimeter is greater than the circle. If a regular hexagon is circumscribed, as in the figure, its area is less than that of the triangle and greater than that of the circle. Similarly, its perimeter is less than that of the triangle and greater than that of the circle. Furthermore, the area and perimeter of the circumscribed polygon are greater than those of the inscribed polygon of the same number of sides, and the area and circumference of the circle lie between the area and perimeter of the inscribed and circumscribed polygons. Therefore, the difference between the area and the perimeter of the circle and those of either the inscribed or circumscribed polygon is less than the difference between the areas or perimeters of these polygons.

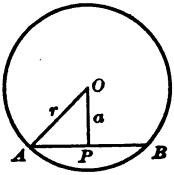
Since this relation remains true however far we carry the process of doubling the number of sides of the polygon:

(1) *The area of the circle is the limit of the area of the inscribed or circumscribed variable polygon.*

(2) *The circumference of the circle is the limit of the perimeter of the inscribed or circumscribed variable polygon.*

PROPOSITION IX. THEOREM

474. *The apothem of the variable regular inscribed polygon formed by successively doubling the number of sides approaches the radius of the circle as a limit.*



Given circle O and AB the side of a variable regular inscribed polygon; also r the radius of O and OP the apothem of the polygon.

To prove that a approaches r as a limit as the sides of the inscribed polygon are successively doubled.

Proof. 1. $a < r$, and $r - a < AP$. Sec. 80, IV

2. But, $AP < AB$, which can be decreased at will.

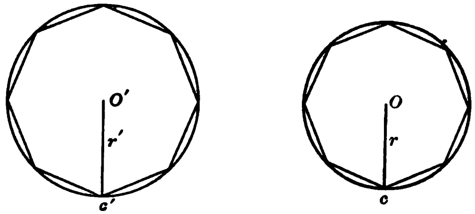
3. $\therefore$ $r - a$ can be made less than any value assigned for AB.

4. $\therefore$ a approaches r as its limit. Sec. 469

475. COROLLARY. *Under the same conditions the square of the apothem approaches the square of the radius as a limit.*

PROPOSITION X. THEOREM

476. *The ratio of the circumference of a circle to its radius is the same for all circles.*



Given any two circles O and O' with circumferences c and c' and radii r and r' respectively.

To prove that $\frac{c}{r}=\frac{c'}{r'}$.

Proof. 1. Inscribe in these circles regular polygons of the same number of sides and having perimeters p and p'.

Then, $\frac{p}{p'}=\frac{r}{r'}$. Sec. 430

2. Double successively the number of sides in each polygon. Then, p and p' have the limits c and c', while r and r' remain constant.

3. Then, $\frac{c}{c'}=\frac{r}{r'}$, Sec. 472

or $cr'=c'r$. Why?

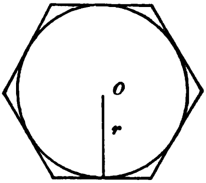
4. $\therefore\ \frac{c}{r}=\frac{c'}{r'}$. Why? Q.E.D.

477. COROLLARY. *The ratio of the circumference of a circle to its diameter is constant; that is, the same for all circles.*

478. COROLLARY. *The circumferences of two circles are in the same ratio as their radii.*

PROPOSITION XI. THEOREM

479. *The area of a circle is equal to half the product of its radius by its circumference.*



Given the circle with radius r, circumference c, and area A.

To prove that $A = \frac{1}{2}rc$.

Proof. 1. Circumscribe a regular polygon of n sides, having a perimeter p and area A'. Then, $A' = \frac{1}{2}rp$. Sec. 432

2. Double successively the number of sides of the circumscribed polygon. Then p approaches c as a limit and r is constant.

3. $\therefore$ $\frac{1}{2}rp$ approaches $\frac{1}{2}rc$ as a limit. Sec. 471
Also, A' approaches A as a limit. Sec. 473
4. $\therefore$ $A = \frac{1}{2}rc$. Sec. 470. Q.E.D.

480. COR. 1. *The area of a circle equals π times the square of the radius, or $A = \pi r^2$.*

For $C = 2\pi r$. Sec. 436
$\therefore$ $A = \frac{1}{2}rc$ becomes $A = \frac{1}{2}r \cdot 2\pi r = \pi r^2$.

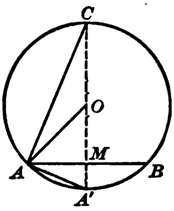
481. COR. 2. *The ratio of the areas of two circles equals the ratio of the squares of their radii.*

482. COR. 3. *The area of a sector is equal to half the product of the radius by the length of its arc.*

For, $\frac{\text{area of sector}}{\text{area of circle}} = \frac{\text{arc of sector}}{\text{circumference}}$.

PROPOSITION XII. PROBLEM

483. *Given the side and radius of a regular inscribed polygon, to express the side of a regular inscribed polygon of double the number of sides.*



Given *AB*, the side of a regular inscribed polygon, and *OA*, the radius of the circle.

To find *AA'* in terms of *r* and *AB*.

Construction. Draw a diameter $A'C \perp AB$. Draw *OA* and *CA*.

Proof. 1. ∴ In the right △ *AOM*, $OM^2 = r^2 - \frac{1}{4}\overline{AB}^2$.

$$\therefore OM = \sqrt{r^2 - \tfrac{1}{4}\overline{AB}^2}.$$

2. $$A'M = r - OM$$

$$= r - \sqrt{r^2 - \tfrac{1}{4}\overline{AB}^2}. \qquad \text{Step (1)}$$

3. $$\overline{AA'}^2 = A'C \cdot A'M$$

$$= 2r\left(r - \sqrt{r^2 - \tfrac{1}{4}\overline{AB}^2}\right). \qquad \text{Step (2)}$$

4. $$\therefore AA' = \sqrt{2r\left(r - \sqrt{r^2 - \tfrac{1}{4}\overline{AB}^2}\right)}$$

$$= \sqrt{r\left(2r - \sqrt{4r^2 - \overline{AB}^2}\right)}.$$

Q.E.D.

484. COR. 1. *If the radius of the circle is a unit of length, then* r = 1, *and* AA' $= \sqrt{2 - \sqrt{4 - \overline{AB}^2}}$.

485. Cor. 2. *If* p *is the perimeter of a regular inscribed polygon of* n *sides, each of which is* s, *then* $p = n\sqrt{2 - \sqrt{4 - s^2}}$.

Proposition XIII. Problem

486. *To approximate the numerical value of the ratio of the circumference of a circle to its diameter.*

Given a circle of circumference c and diameter d.

To calculate the numerical value of $\frac{c}{d}$, or π.

Solution. 1. Let s_6 denote the side of a regular inscribed polygon of 6 sides, s_{12} denote one of 12 sides, and so on. Also let p_6 denote the perimeter of an inscribed polygon of 6 sides, p_{12} denote the perimeter of one of 12 sides, and so on.

2. Since $s_6 = 1$, then $p_6 = 6$. Sec. 421

$s_{12} = \sqrt{2 - \sqrt{4 - 1}} = .51763809.$ Sec. 483

$\therefore p_{12} = 6.21165708.$

$s_{24} = \sqrt{2 - \sqrt{4 - (.51763809)^2}} = .26105238.$

$\therefore p_{24} = 6.26525722.$

$s_{48} = \sqrt{2 - \sqrt{4 - (.26105238)^2}} = .13080626.$

$\therefore p_{48} = 6.27870041.$

Continuing in this way,

$s_{384} = \sqrt{2 - \sqrt{4 - (.03272346)^2}} = .01636228.$

$\therefore p_{384} = 6.28311544.$

Taking p_{384} as the approximate value of c,

$c = 6.28311544.$

4. But $\pi = \frac{1}{2}c$, when $r = 1$. Since $2\pi r = c$. Sec. 436

$\therefore \pi = \frac{1}{2}$ of 6.28311544, or 3.1416 nearly.

Note. — Like $\sqrt{2}$ and many other numbers, the number π cannot be expressed exactly in decimal form. The above calculation serves merely to show how an approximate value for π might be computed in elementary geometry. The methods used by modern mathematicians in the computation of approximate values for π are based upon quite different considerations. See pp. 279, 280.

EXERCISES

1. Compute the length of the side of a regular inscribed polygon of 12 sides when $r = 1$.

Suggestion. What is the length of the side of the regular inscribed hexagon?

2. Compute the side of the regular inscribed octagon when $r = 1$.

3. Compute the perimeter of a regular circumscribed octagon when $r = 1$.

VI. SUMMARY OF IMPORTANT FORMULAS

487. Notations. In general, small letters denote lines or linear values, and capital letters represent points or areas. The following are the abbreviations usually met:

a, b, c = sides of $\triangle ABC$.
a = apothem, sometimes altitude.
a' = projection of a.
a^2 = area of square of side a.
A = area of $\triangle ABC$.
b = base.
b' is used when there are two bases.
c = circumference.
d = diameter, or diagonal.
h = height, sometimes altitude.
h_a, h_b, h_c = the altitudes of $\triangle ABC$ to sides a, b, c.
l = length.
m = median.
m_a, m_b, m_c = medians of $\triangle ABC$ to sides a, b, c.
p = perimeter.
P = area of polygon.
$\pi = 3.1416$, approximately $3\frac{1}{7}$.
r = radius.
s = side of regular polygon, or semiperimeter of $\triangle ABC = \frac{1}{2}(a + b + c)$.

488. Important Formulas.

1. Relation between sides of right triangle, $a^2 = b^2 + c^2$. Sec. 368
2. Relation between sides of any triangle, $a^2 + b^2 + 2\,ab' = c^2$. Sec. 376
3. Circumference of circle, $c = 2\,\pi r = \pi d$. Sec. 436
4. Altitude of equilateral triangle, $h = \frac{1}{2}\,s\sqrt{3}$. Ex. 27, p. 229
5. Diagonal of a square, $d = a\sqrt{2}$. Sec. 370
6. Side of regular inscribed polygon of double the number of sides $\sqrt{r'(2\,r - \sqrt{4\,r^2 - s^2})}$. Sec. 483
7. Area of a rectangle, bh. Sec. 347
8. Area of a square, a^2. Sec. 347
9. Area of a parallelogram, bh. Sec. 355
10. Area of a triangle, $\frac{1}{2}\,bh$, or $\sqrt{s(s-a)(s-b)(s-c)}$. Sec. 356, Ex. 5, p. 232
11. Area of an equilateral triangle, $\frac{1}{4}\,b^2\sqrt{3}$. Ex. 29, p. 229
12. Area of a trapezoid, $\frac{1}{2}\,h\,(b + b')$. Sec. 364
13. Area of a regular polygon, $\frac{1}{2}\,ap$. Sec. 432
14. Area of circle, $\frac{1}{2}\,rc = \pi r^2$. Sec. 437
15. Area of a sector, $\frac{\text{central angle}}{360} \cdot \pi r^2$. Sec. 482

INDEX

The numbers denote the pages where the topics are found.

Acute angle, 14.
Adjacent angles, 11.
Alternation, proportion by, 174.
Altitude, 50, 80.
Analytic proof, 100.
Angle, 9.
  acute, 14.
  complement of, 14.
  exterior, 11, 54, 65.
  inscribed, 125.
  oblique, 14.
  obtuse, 14.
  plane, 10.
  reflex, 11.
  right, 13.
  sides of, 9.
  size of, 10, 15.
  straight, 11.
  supplement of, 14.
  vertex of, 9.
Angles, 11.
  adjacent, 11.
  alternate exterior, 65.
  alternate interior, 65.
  complementary, 14.
  corresponding, 62.
  equal, 10.
  exterior, 11.
  generation of, 10.
  interior, 11.
  of a polygon, sum of, 87.
  of a triangle, sum of, 74.
  supplementary, 14.
  vertical, 14, 28.
Apothem, 258.
Arc, 21.
  major, 110.
  minor, 110.
Archimedes, 272.
Area, of circle, 21, 271, 303.
  of irregular polygon, 222.
Area of parallelogram, 219.
  rectangle, 217.
  trapezoid, 222.
  triangle, 220.
Axioms, list of, 24.
Axis of symmetry, 281.

Base, 80.
Bisector, 8, 11.
Broken line, 2.

Center, of circle, 20.
  of regular polygon, 258.
  of symmetry, 281.
Chord, 21.
  of contact, 129.
Circle, 20, 108, 170.
  arc of, 21.
  area of, 21, 271, 303.
  as a limit, 300.
  as a locus, 95.
  center of, 20.
  chord of, 21.
  circumference of, 21.
  circumscribed, 124.
  common chord, 115.
  diameter of, 21.
  inscribed, 124.
  radius of, 21.
  secant to, 62, 120.
  sector of, 270.
  segment of, major, minor, 124.
  tangent to, 120.
Circles, concentric, 120.
  escribed, 152.
  tangent, 121.
Circumference, 21.
Circumscribed circle, 124.
Circumscribed polygon, 124.
Commensurable magnitudes, 171.
Common measure, 171.

Common tangents, 159, 160.
Complement, 14.
Composition, proportion by, 174.
Concave polygon, 87.
Concentric circles, 120.
Concurrent lines, 90.
Congruent, figures, 31.
Constant, 299.
Continuity, principle of, 298.
Converse propositions, 111.
Converse theorems, law of, 111.
Convex polygon, 87.
Corollary, 27.
Corresponding angles, 62.
Corresponding parts, 33.
Curved line, 2.

Decagon, 86.
Degree, 16.
Demonstration, 29.
Descartes, 168.
Determinate cases, 157.
Diagonal, 79, 188.
Diameter, 21.
Difference, of angles, 12.
  arcs, 110.
Directed segments, 293.
Discussion of a problem, 157.
Distance, from point to line, 51.
Division, harmonic, 205.
Drawing figures, 21.

Equal angles, 10.
Equiangular polygon, 87, 250.
Equiangular triangle, 39.
Equilateral polygon, 87, 250.
Equilateral triangle, 20.
Equivalent figures, 221.
Escribed circles, 152.
Euclid, 61.
Exterior angles, 11, 54, 65.
Extreme and mean ratio, 205.
Extremes, 173.

Figure, 4.
  geometric, 4.
  plane, 4.
  symmetric, 281.
Figures, equivalent, 221.
  isoperimetric, 283.
  symmetric, 281.
Formulas, 307.
Fourth proportional, 173.

Gauss, 265.
Generation of angles, 10.
Geometric figure, 4.
Geometry, 1, 4.

Harmoinc division, 205.
Hexagon, 86.
Historical notes, 27, 61, 107, 168, 212, 248, 265, 272, 279.
Homologous parts, 33.
Hypotenuse, 20, 226.
Hypothesis, 29.

Impossible cases, 157.
Incommensurable magnitudes, 171.
Incommensurable ratio, 172.
Indeterminate cases, 157.
Indirect proof, 103.
Inscribed angle, 125.
Inscribed circle, 124.
Inscribed polygon, 125.
Instruments, 21.
Interior angles, 11, 65.
Inversion, proportion by, 174.
Isoperimetric polygons, 283.
Isosceles trapezoid, 79.
Isosceles triangle, 20.

Limit, 299.
Limits, principle of, 299.
Line, 2.
  broken, 2.
  curved, 2.
  of centers, 115.
  straight, 2.
Lines, concurrent, 90.
  multiplication of, 8.
  parallel, 62.
  perpendicular, 14.
  transversal of, 62.
Loci, solutions by, 97.
Locus, 94.
  proof of, 94.

Magnitudes, 171.
  bisectors of, 8.
  commensurable, 171.
  constant, 299.
  generation of, 3, 4.

Magnitudes, incommensurable, 171.
  variable, 299.
Maxima and minima, 283.
Maximum, 283.
Mean proportional, 173.
Means, 173, 201.
Measure, 170.
  common, 171.
  unit of, 214.
Median, 34.
Methods, of solution, 157.
  of proof, 100.
Minimum, 283.

Need for proof, 22.
Negative quantities, 293.

Oblique angle, 14.
Oblique lines, 1.
Obtuse angle, 14.
Octagon, 86.

Parallel lines, 62.
Parallelogram, 79.
Pentagon, 86.
Perigon, 16.
Perimeter, 20, 79.
Perpendicular, 45.
Pi (π), 279.
  value of, 279.
Plane, 3.
  angle, 10.
  geometry, 4.
Plato, 107.
Point, 2.
  of contact, 120.
Polygon, 86.
  apothem of regular, 258.
  area of, 222, 268.
  center of regular, 258.
  circumscribed, 124.
  concave, 87.
  convex, 87.
  diagonal of, 86.
  equiangular, 250.
  equilateral, 250.
  inscribed, 125.
  perimeter of, 86.
  radius of regular, 258.
  regular, 250.
Polygons, classified, 86, 250.
Polygons, equiangular, 87.
  equilateral, 87.
  isoperimetric, 283.
  similar, 187.
Positive quantities, 295, 297.
Postulate, 25.
  of parallels, 63.
Postulates, list of, 26.
Principle of limits, 299.
Problem, 26.
  how to attack, 156.
Product of lines, 25.
Projection, 229.
Proof, methods of, 23, 100.
  nature of, 29.
  necessity for, 22.
Proportion, 173.
Proportional, fourth, 173.
  mean, 173.
Proposition, 27.
Pythagoras, 248.
Pythagorean theorem, 226, 248.

Quadrilateral, 79.
Quadrilaterals classified, 79.

Radius, 21.
  of regular polygon, 258.
Ratio, 172.
  extreme and mean, 205.
  incommensurable, 172.
  of similitude, 188.
Rectangle, 79.
Reductio ad absurdum, 103.
Reflex angle, 11.
Regular polygon, 87, 250.
Rhombus, 79.
Right angle, 13.
Right triangle, 20.

Scalene triangle, 20.
Secant, 62, 120.
Sector, 270.
Segment, of a circle, 124.
  of a line, 5.
Semicircle, 109.
Similar polygons, 187.
Similitude, ratio of, 188.
Square, 79.
Straight angle, 11.
Straight line, 2.

Subtend, 21, 109.
Superposition, 31.
Supplement, 14.
Surface, 3, 4.
Symmetric figures, 281.
Symmetry, 281.
Synthetic proof, 100.

Tangent, 120.
Tangent circles, 121.
Thales, 27.
Theorem, 26.
Transversal, 62.
Trapezoid, 79.
Triangle, 19.
Triangle, altitude of, 50.
  area of, 200.
  base of, 80.
  equilateral, 20.
  isosceles, 20.
  right, 20.
  scalene, 20.
  sides of, 19.
  vertices of, 19.

Unit of measure, 214.

Variable, 299.
Vertex of angle, 9.
Vertical angles, 14, 28.

Teachers College. N. Y. C
3/22/83